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Contents

1 Introduction

Our thesis is a contribution to the present knowledge of the dynamics of numerical procedures applied to continuous dynamical system (DS). The emphasis has been made on two topics discussed in two independent chapters.

In the first chapter we explored parameterized Lipschitzian and Carathéodorian semi-linear differential inclusions in Banach spaces with exponentially dichotomous linear parts. Under additional assumptions, we proved the existence and uniqueness of quasibounded solutions. Then the analogy of the stable and unstable sets corresponding to these quasibounded solutions were defined and it turned out that they are the graphs of suitable multifunctions. We also introduced and studied solutions corresponding to more general weighted selector spaces. We discussed hierarchy like in [3] and a special type of their independence. Chapter 1 was concluded with presenting some criteria on the existence of hyperbolic exponential dichotomy on $\mathbb{R}$. These sufficient conditions were derived for constant matrices on a finite dimensional $\mathbb{C}^n$, for a class of infinite matrices on complex ℓ_p spaces and finally for some non-autonomous periodic ODE's also on ℓ_p.

After that, Chapter 2 was devoted to the precise analytical derivation of the numerical/discretized Poincaré map $\mathcal{P}_m$ of an ordinary differential equation possessing a periodic orbit. We have been motivated by papers [33, 64], where numerical tools were used for computing the Poincaré map. Our goal was to give a precise analytical meaning of $\mathcal{P}_m$ and to establish error bounds for the difference $|\mathcal{P} - \mathcal{P}_m|$ and its various differentials. Our approach used the method of a moving orthonormal system (introduced rigorously in [32] and then applied successfully in [6, 8, 59]) and the Newton–Kantorovich type theorem (cf. [37,47,67]). In the end of Chapter 2 we applied the previously established properties of $\mathcal{P}_m$. In Section 2.4 under the nondegeneracy of γ we detected a third interesting curve specially related to the discrete dynamics. Namely the set of those points which are invariant in a proper sense under the action of $\mathcal{P}_m$. We also gave a short remark about the spectral property of this curve.

In this overview at first we briefly define the main notions needed to be able to state the new results of the thesis. Second, we state these theorems and add some comments to them. Rigorous proofs of the foregoing statements and even more can be found in the thesis.

2 Prerequisites for results in Chapter 1

We suppose that X is a real Banach space with a norm $|\cdot|$ and we denote by X_1 the closed unit ball in X. Further, by $\mathcal{B}(X)$ let us designate the Banach space of bounded and linear operators $L : X \to X$.

Measure theory: We say that an interval $I \subset \mathbb{R}$ of arbitrary type is *positive* if $|I| > 0$ for its (Lebesgue) measure (the case $+\infty$ is also involved). Let us have a positive interval I. The function $f : I \to X$ is *strongly measurable* (*s. m.*) if the range $f(I)$ is separable and f is (Borel) measurable (f is measurable if the pre-image $f^{-1}(B)$ is a Borel set for all Borel sets $B \subset X$). Further f is *simple* if it has only finitely many values and is strongly measurable. A function $f : I \times X \to X$ has a *Carathéodory property* if, on one hand, $f(t, \cdot) : X \to X$ is continuous for all fixed $t \in I$ and, on the other hand, $f(\cdot, x) : I \to X$ is s. m. for all fixed $x \in X$. We denote the set of these functions by $\mathcal{CAR}(I, X)$. We suppose that the reader has been acquainted with the theory of Lebesgue integrals. The brief definition of Bochner integrals using Lebesgue integrals is the following one: a s. m. function $f : I \to X$ is *Bochner integrable* (or simply *integrable*) if the norm function $|f| : I \to \mathbb{R}$ defined as $|f|(t) := |f(t)|$ is Lebesgue integrable. The function f is called *locally integrable* if it is s. m. on I and integrable over compact subintervals of I.

For an integrable simple function $f = \sum_{j=1}^{k} \alpha_j \chi_{I_j}$, where $\alpha_j \in \mathbb{R}, I_j \subset I$ are measurable and χ_{I_j} is the characteristic function of the set I_j, we define the Bochner integral as $\int_I f \mathrm{d}t := \sum_{j=1}^{k} a_j |I_j|$. For an arbitrary integrable function one can find simple integrable functions f_n such that $f = \lim_{n\to\infty} f_n$, $|f_n(t)| \leq |f(t)|$ (see [11, Appendix E]). Then the well-known Lebesgue's Dominated Convergence Theorem for real-valued functions implies the well-definiteness of $\int_I f \mathrm{d}t := \lim_{n\to\infty} \int_I f_n \mathrm{d}t$.

Solution concepts, selectors: Let $J, I, J \subset I$ are positive intervals and $\mathcal{M}$ is a topological (mainly metric) space. Let $f : I \times X \times \mathcal{M} \to X$ satisfies $f(\cdot, \cdot, y) \in \mathcal{CAR}(I, X)$ for all $y \in \mathcal{M}$. A continuous function $\lambda : J \to X$ is said to be a *solution* of the ODE $\dot{x} = f(t, x, y)$ at the parameter value $y \in \mathcal{M}$ if the function $f(\cdot, \lambda(\cdot), y) : J \to X$ is locally integrable and $\lambda(t) - \lambda(s) = \int_s^t f(\tau, \lambda(\tau), y) \mathrm{d}\tau$ holds for all $s, t \in J$. In addition we say that λ satisfies the *initial condition* $x(t_0) = x_0$ for some fixed values $t_0 \in I, x_0 \in X$ if $t_0 \in J$ and $\lambda(t_0) = x_0$.

For a positive J let us define selector spaces $\mathsf{H}(J) := \{h : J \to X \ : h$ is s. m. and $|h|_{J,\infty} < \infty\}$ with $|h|_{J,\infty} = \sup_{t\in J} |h(t)|$. Then $\mathsf{H}(J)$ endowed with the norm $|\cdot|_{J,\infty}$ turns into a Banach space. For simplicity we introduce also $\mathsf{H} := \mathsf{H}(\mathbb{R})$, $|\cdot|_\infty := |\cdot|_{\mathbb{R},\infty}$, $\mathsf{H}^\pm_\tau := \mathsf{H}(\mathbb{R}^\pm_\tau)$, $|\cdot|^\pm_\tau := |\cdot|_{\mathbb{R}^\pm_\tau,\infty}$, where $\mathbb{R}^+_\tau := [\tau, \infty)$ and $\mathbb{R}^-_\tau := (-\infty, \tau]$ for $\tau \in \mathbb{R}$. We will refer to the elements of $\mathsf{H}(J)$ as *selectors*.

Consider positive intervals J, I such that $J \subset I$. A continuous $\lambda : J \to X$ is called a *solution* of the inflated differential equation (IDE) $\dot{x} \in f(t, x, X_1)$ corresponding to the selector $h \in \mathsf{H}(J)_1$ if λ is a solution of $\dot{x} = F(t, x, h(t))$. In addition we say that λ satisfies the initial condition $x(t_0) = x_0$ for $t_0 \in I, x_0 \in X$ if we have $t_0 \in J$ and $\lambda(t_0) = x_0$.

Exponential dichotomy and quasiboundedness: We say that the equation $\dot{x} = A(t)x$ possesses an *exponential dichotomy* on the positive interval I and for a locally integrable $A : I \to \mathcal{B}(X)$ if there are constants $K \geq 1$, $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}$ and a *projection* $P \in \mathcal{B}(X)$ (means that $P^2 = P$) such that $\left|\Phi(t, 0) \circ P^+ \circ \big(\Phi(0, s)\big)^{-1}\right|_{\mathcal{B}(X)} \leq K\mathrm{e}^{\alpha(t-s)}$, for $t \geq s, t, s \in I$ and $\left|\Phi(t, 0) \circ P^- \circ \big(\Phi(0, s)\big)^{-1}\right|_{\mathcal{B}(X)} \leq K\mathrm{e}^{\beta(t-s)}$, $t \leq s, t, s \in I$, where $P^+ := P$ and $P^- := \mathbb{I}_X - P^+$ and Φ is the *evolution operator* of $\dot{x} = A(t)x$ (that is the operator solution of $\dot{Y} = A(t)Y, Y(s) = \mathbb{I}_X, Y(\cdot) \in \mathcal{B}(X)$). We denote by $\mathcal{E}_{\alpha,\beta}(I)$ the set of all locally integrable $A : \mathbb{R} \to L(X)$ for which $\dot{x} = A(t)x$ possesses an exponential dichotomy on I. Furthermore we introduce notations $P^\pm(t) := \Phi(t, 0) \circ P^\pm \circ \big(\Phi(0, t)\big)^{-1}$, $\mathbb{P}^\pm_t := P^\pm(t)(X)$.

We say that the interval I is unbounded to the left if I is one of the interval types $(-\infty, a)$, $(-\infty, a]$, $\mathbb{R}$ and similarly we use the term "unbounded to the right". Assume that I is unbounded to the left (resp. to the right). Let $g : I \to X$ be an arbitrary function and $\gamma \in \mathbb{R}$. We say that g is γ^-*-quasibounded* (resp. γ^+*-quasibounded*; we use the abbreviation q. b.) if $\|g\|^-_{\tau,\gamma} < \infty$ (resp. $\|g\|^+_{\tau,\gamma} < \infty$) for some $\tau \in I$, where $\|g\|^-_{\tau,\gamma} := \sup_{t\in\mathbb{R}^-_\tau} |g(t)|\mathrm{e}^{-\gamma t}$, (resp. $\|g\|^+_{\tau,\gamma} := \sup_{t\in\mathbb{R}^+_\tau} |g(t)|\mathrm{e}^{-\gamma t}$). In the peculiar $I = \mathbb{R}$ case we say that g is γ*-q. b.* if $\|g\|_\gamma := \sup_{t\in\mathbb{R}} |g(t)|\mathrm{e}^{-\gamma t} < \infty$.

3 New results in Chapter 1

We always assume that $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R}), \alpha < \beta$. With an elementary transformation (Lemma 1.17 of the thesis) we were able to prove the following generalization of [12, Theorem 3].

Theorem 1 (Theorem 1.18 of the thesis)**.** *Assume that we have functions* $f : \mathbb{R} \times X \to X, g : \mathbb{R} \times X \times X_1 \to X$ *and a constant* $\gamma \in (\alpha, \beta)$ *such that*

(i) Smoothness: $f(\cdot, x), g(\cdot, x, u) : \mathbb{R} \to X$ *are s. m. for all* $x \in X, u \in X_1$ *and* $f(t, \cdot) : X \to X, g(t, \cdot, \cdot) : X \times X_1 \to X$ *are continuous for all* $t \in \mathbb{R}$,

(ii) Quasiboundedness: $\|f(t,0)\|_\gamma < \infty, \|g(t,0,0)\|_\gamma < \infty$,

(iii) Lipschitz condition: *there are constants* L_1, L_2, L_3 *such that* $|f(t,x_1) - f(t,x_2)| \leq L_1|x_1 - x_2|$ *and*

$$|g(t,x_1,u_1) - g(t,x_2,u_2)| \leq L_2|x_1 - x_2| + L_3 \mathrm{e}^{\gamma t}|u_1 - u_2| \tag{L}$$

are valid for all $t \in \mathbb{R}, x_1, x_2 \in X, u_1, u_2 \in X_1$,

(iv) $K(L_1 + L_2)\kappa_{\alpha-\gamma,\beta-\gamma} < 1$.

Then for every $h \in \mathsf{H}_1$ *there exists a unique* γ*-q. b. solution* $\Gamma_\gamma(\cdot, h) : \mathbb{R} \to X$ *of the problem* $\dot{x} \in A(t)x + f(t,x) + g(t,x,X_1)$ *corresponding to the selector* h. *In addition the mapping* $\Gamma_\gamma : \mathbb{R} \times \mathsf{H}_1 \to X$ *is continuous and Lipschitz in the second variable.*

In a setting of Theorem 1 we put down an important set of initial positions of the quasibounded solutions $S^\gamma_{\tau,\varepsilon} := \{\Gamma(\tau, h) \ : \ h \in \mathsf{H}_\varepsilon\}$, $\varepsilon \in [0,1], \tau \in \mathbb{R}$. Let us introduce $\mathsf{H}_\varepsilon := \{h \in \mathsf{H} \ : \ |h|_\infty \leq \varepsilon\}$, and $\mathsf{H}^\pm_{\tau,\varepsilon} := \{h : \mathbb{R}^\pm_\tau \to X$ is s. m. and $|h|^\pm_\tau \leq \varepsilon\}$, where $\tau \in \mathbb{R}, \varepsilon \in [0,1]$. Note that $\mathsf{H}_\varepsilon, \mathsf{H}^\pm_{\tau,\varepsilon}$ are complete metric spaces with corresponding metrics derived naturally from norms $|\cdot|_\infty, |\cdot|^\pm_\tau$.

Denote by $\lambda(\cdot, t_0, x_0, h)$ the unique solution of $\dot{x} = A(t)x + f(t,x) + g(t,x,h(t))$, $x(t_0) = x_0$ for a triple $(t_0, x_0, h) \in \mathbb{R} \times X \times \mathsf{H}_1$ or $(t_0, x_0, h^\pm) \in \mathbb{R} \times X \times \mathsf{H}^\pm_{\tau,1}$ (for a formal ambiguity in this notation, see the discussion around the equation (1.16) in the thesis).

For arbitrary functions $f : D_f \to X, g : D_g \to X$ we write $f \subset g$ if $D_f \subset D_g$ and $g|_{D_f} = f$.

Now we define the following sets (these are the analogous of the stable and unstable set from the hyperbolic setting)

$$\begin{aligned} M^{s,\gamma}_{\tau,\varepsilon} := \{\xi \in X \ : \ &\exists h^+ \in \mathsf{H}^+_{\tau,\varepsilon}, \exists h \in \mathsf{H}_\varepsilon, h^+ \subset h \text{ such that} \\ &\lim_{t\to\infty} \|\lambda(t,\tau,\xi,h^+) - \Gamma_\gamma(t,h)\|\mathrm{e}^{-\gamma t} = 0\}, \\ M^{u,\gamma}_{\tau,\varepsilon} := \{\xi \in X \ : \ &\exists h^- \in \mathsf{H}^-_{\tau,\varepsilon}, \exists h \in \mathsf{H}_\varepsilon, h^- \subset h \text{ such that} \\ &\lim_{t\to-\infty} \|\lambda(t,\tau,\xi,h^-) - \Gamma_\gamma(t,h)\|\mathrm{e}^{-\gamma t} = 0\}. \end{aligned}$$

We can state the following generalization of [12, Theorem 4].

Theorem 2 (Theorem 1.19 of the thesis)**.** *Suppose all the assumptions of Theorem 1 and fix* $\tau \in \mathbb{R}, \varepsilon \in [0,1]$. *Then there are Lipschitz continuous functions* $w^{s,\gamma} : \mathbb{P}^+_\tau \times \mathsf{H}^+_{\tau,\varepsilon} \to \mathbb{P}^-_\tau$,

$w^{u,\gamma} : \mathbb{P}_\tau^- \times \mathsf{H}_{\tau,\varepsilon}^- \to \mathbb{P}_\tau^+$ *such that*

$$\begin{aligned} M_{\tau,\varepsilon}^{s,\gamma} &= \{\xi \in X \; : \; \exists h^+ \in \mathsf{H}_{\tau,\varepsilon}^+ \; : \; \|\lambda(\cdot,\tau,\xi,h^+)\|_{\tau,\gamma}^+ < \infty\} \\ &= \{\xi^+ + w^{s,\gamma}(\xi^+,h) \; : \; \xi^+ \in \mathbb{P}_\tau^+, h \in \mathsf{H}_{\tau,\varepsilon}^+\}, \\ M_{\tau,\varepsilon}^{u,\gamma} &= \{\xi \in X \; : \; \exists h^- \in \mathsf{H}_{\tau,\varepsilon}^- \; : \; \|\lambda(\cdot,\tau,\xi,h^-)\|_{\tau,\gamma}^- < \infty\} \\ &= \{\xi^- + w^{u,\gamma}(\xi^-,h), : \; \xi^- \in \mathbb{P}_\tau^-, h \in \mathsf{H}_{\tau,\varepsilon}^-\}. \end{aligned}$$

Exact Lipschitz constants were found in the thesis.

These two theorems are novelty mainly because we replaced the hyperbolic assumption $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R})$, $\alpha < 0 < \beta$ by a general one $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R}), \alpha < \beta$. We might note that the stated results are definitely not shockingly new, rather a systematic and surprisingly easy generalization of the previously known theory. One might be curious about the necessity of the condition (L). It was possible to avoid it by introducing new selector spaces. We do not explain the details here, one should go through the detailed Remarks 1.8, 1.9 of the thesis.

Section 1.2 of the thesis is concluded by the answers of the two interesting questions

Q1: Under which conditions are we able to prove the independence of Γ_γ on γ?

Q2: What relations should we expect between various stable/unstable-like sets if the linear part possesses exponential dichotomy on $\mathbb{R}$ corresponding to more then one, properly linked projection?

A partial answer to question **Q1:**

Theorem 3 (Theorem 1.20 of the thesis)**.** *Let us have $\alpha < \alpha_1 < \beta_1 < \beta$ and functions $f : \mathbb{R} \times X \to X$, $g : \mathbb{R} \times X \times X_1 \to X$ such that*

(i) Smoothness: *$f(\cdot,x), g(\cdot,x,u) : \mathbb{R} \to X$ are s. m. for all $x \in X, u \in X_1$ and $f(t,\cdot) : X \to X, g(t,\cdot,\cdot) : X \times X_1 \to X$ are continuous for all $t \in \mathbb{R}$,*

(ii) Upper bound: *there are constants $M_1, M_2 \geq 0$ such that*

$$|f(t,0)| \leq M_1\eta(t), \quad |g(t,0,0)| \leq M_2\eta(t), \qquad t \in \mathbb{R},$$

where $\eta(t) := \min\{e^{\alpha_1 t}, e^{\beta_1 t}\}$,

(iii) Lipschitz condition: *there are constants $L_1, L_2, L_3 \geq 0$ such that*

$$\begin{aligned} |f(t,x_1) - f(t,x_2)| &\leq L_1|x_1 - x_2|, \\ |g(t,x_1,u_1) - g(t,x_1,u_2)| &\leq L_2|x_1 - x_2| + L_3\eta(t)|u_1 - u_2| \end{aligned}$$

are valid for all $t \in \mathbb{R}, x_1, x_2 \in X, u_1, u_2 \in X_1$, *(iv) for a constant* $\theta := \max\left\{\kappa_{\alpha-\alpha_1,\beta-\alpha_1}, \kappa_{\alpha-\beta_1,\beta-\beta_1}\right\}$ *we have* $K(L_1+L_2)\theta < 1$.

Then Γ_γ *from Theorem 1 is well-defined for* $\gamma \in [\alpha_1, \beta_1]$ *and independent from* γ *– that is* $\Gamma_{\gamma_1} = \Gamma_{\gamma_2}$ *for all* $\gamma_1, \gamma_2 \in [\alpha_1, \beta_1]$.

Focusing now on the question **Q2** let us have for $i = 1, \cdots, n, n \geq 2$ projections $P_i \in \mathcal{B}(X)$ (that is $P_i^2 = P_i$) and $K_i \geq 1, \alpha_i, \beta_i, \gamma_i \in \mathbb{R}$. Suppose that $\alpha_i < \gamma_i < \beta_i$, $i = 1, \cdots, n$ and $\beta_i \leq \alpha_{i+1}, i = 1, \cdots, n-1$. Set $P_i^+ := P_i, P_i^- := \mathbb{I} - P_i$ and assume the following *hierarchy* of the projector ranges

$$P_i^+(X) \subset P_{i+1}^+(X), \quad P_i^-(X) \supset P_{i+1}^-(X), \qquad i = 1, \cdots, n-1. \tag{H}$$

Further, suppose $A \in \mathcal{E}_{\alpha_i,\beta_i}(\mathbb{R}) = \mathcal{E}_{\alpha_i,\beta_i}(\mathbb{R}; P_i, K_i)$ that is $\left|\Phi(t,0)P_i^+\Phi(0,s)\right| \leq K_i e^{\alpha_i(t-s)}$ for $t \geq s$ and $\left|\Phi(t,0)P_i^-\Phi(0,s)\right| \leq K_i e^{\beta_i(t-s)}$ for $t \leq s$. Introduce moreover $\eta^*(t) := \min_{i=1,\cdots,n}\{e^{\gamma_i t}\}$ and $\theta^* := \max_{i=1,\cdots,n}\{\kappa_{\alpha_i-\gamma_i,\beta_i-\gamma_i}\}$. Under these assumptions we can state the following result.

Theorem 4 (Theorem 1.21 of the thesis)**.** *Let* $f : \mathbb{R} \times X \to X$, $g : \mathbb{R} \times X \times X_1 \to X$ *are such that*

(i) Smoothness: $f(\cdot, x), g(\cdot, x, u) : \mathbb{R} \to X$ *are s. m. for all* $x \in X, u \in X_1$ *and* $f(t, \cdot) : X \to X, g(t, \cdot, \cdot) : X \times X_1 \to X$ *are continuous for all* $t \in \mathbb{R}$,

(ii) Upper bound: *there is a constants* $M \geq 0$ *such that* $|f(t,0)| \leq M\eta^*(t)$, $|g(t,0,0)| \leq M\eta^*(t)$, *for* $t \in \mathbb{R}$,

(iii) Lipschitz condition: *there are constants* L_1, L_2, L_3 *such that*

$$|f(t,x_1) - f(t,x_2)| \leq L_1|x_1 - x_2|$$
$$|g(t,x_1,u_1) - g(t,x_1,u_2)| \leq L_2|x_1 - x_2| + L_3\eta^*(t)|u_1 - u_2|$$

are valid for all $t \in \mathbb{R}, x_1, x_2 \in X, u_1, u_2 \in X_1$,

(iv) we have $K(L_1 + L_2)\theta^* < 1$.

Then $\Gamma_{\gamma_i}, M^{s,\gamma_i}_{\tau,\varepsilon}, M^{u,\gamma_i}_{\tau,\varepsilon}$ *from Theorems 1 and 2 concerning the IDE* $\dot{x} \in A(t)x + f(t,x) + g(t,x,X_1)$ *are well-defined and the following inherited (from* (H)*) hierarchy is valid*

$$M^{s,\gamma_i}_{\tau,\varepsilon} \subset M^{s,\gamma_{i+1}}_{\tau,\varepsilon}, \qquad M^{u,\gamma_i}_{\tau,\varepsilon} \supset M^{u,\gamma_{i+1}}_{\tau,\varepsilon}, \qquad i = 1, \cdots, n-1.$$

As far as we see question **Q1** has not been investigated yet. The hierarchy of integral manifolds for non-autonomous systems without inflation and with a bit restrictive $f(t,0) = 0$ was brilliantly presented in [3, 4]. The previously developed theory made the proofs of Theorems 3 and 4 very easy, in fact they are only the consequences of Theorems 1 and 2 in an adequate framework.

Chapter 1 was finished by some comments about hyperbolic exponential dichotomy on complex spaces. For the finite dimensional case the proper use of Neumann's Inversion Lemma (c.f. Lemma 1.22 of the thesis) yields

Theorem 5 (Theorem 1.23 of the thesis)**.** *Consider an $n \times n$ complex valued matrix $A = (a_{ij})_{i,j=1}^n$. Fix $\lambda \in \mathbb{C}$, suppose $a_{ii} \neq \lambda, i = 1, \cdots, n$ and set $d := \max_{1 \le i \le n} \{|\lambda - a_{ii}|^{-1}\}$, $A_\lambda := \lambda \mathbb{I} - A$. Then the following statements hold*

1. If $\eta_1 := \max_{1 \le i \le n} \left\{ \sum_{j=1, j \neq i}^n \frac{|a_{ji}|}{|\lambda - a_{jj}|} \right\} < 1$, then A_λ is invertible and $\|A_\lambda^{-1}\|_1 \le \frac{d}{1-\eta_1}$.

2. If $\eta_\infty := \max_{1 \le i \le n} \left\{ \frac{\sum_{j=1, j \neq i}^n |a_{ij}|}{|\lambda - a_{ii}|} \right\} < 1$, then A_λ is invertible and $\|A_\lambda^{-1}\|_\infty \le \frac{d}{1-\eta_\infty}$.

3. If $\tau_p := \sum_{i=1}^n \frac{\left(\sum_{j=1, j \neq i}^n |a_{ij}|^q\right)^{p/q}}{|\lambda - a_{ii}|^p} < 1$, for some $p > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$, then A_λ is invertible and $\|A_\lambda^{-1}\|_p \le \frac{d}{1 - \sqrt[p]{\tau_p}}$.

A nearly straightforward consequence was the following theorem on the infinite dimensional ℓ_p spaces.

Theorem 6 (Theorem 1.24 of the thesis)**.** *Consider an infinite matrix A defined formally as $(Ax)_i := \sum_{j=i-s}^{j+s} a_{ij} x_j, i \in \mathbb{Z}$ and $s \in \mathbb{N}$ for a bounded sequence $\{a_{ij}\}_{i,j \in \mathbb{Z}}^{|i-j| \le s}$. Suppose $\lambda \in \mathbb{C}$ and $\omega := \inf_{i \in \mathbb{Z}} |\lambda - a_{ii}| > 0$ then the following statements hold*

1. If $\eta_1 := \sup_{i \in \mathbb{Z}} \sum_{j=i-s, j \neq i}^{j=i+s} \frac{|a_{ji}|}{|\lambda - a_{jj}|} < 1$, then A_λ is invertible in ℓ_1 and $\|A_\lambda^{-1}\|_1 \le (\omega(1 - \eta_1))^{-1}$.

2. If $\eta_\infty := \sup_{i \in \mathbb{Z}} \sum_{j=i-s, j \neq i}^{i+s} \frac{|a_{ij}|}{|\lambda - a_{ii}|} < 1$, then A_λ is invertible in ℓ_∞ and $\|A_\lambda^{-1}\|_\infty \le (\omega(1 - \eta_\infty))^{-1}$.

3. If $\tau_p := \sup_{i \in \mathbb{Z}} \sum_{k=i-s}^{i+s} \frac{\left(\sum_{j=k-s, j \neq k}^{k+s} |a_{kj}|^q\right)^{p/q}}{|\lambda - a_{kk}|^p} < 1$, for some $p \in (1, \infty)$, where $\frac{1}{p} + \frac{1}{q} = 1$, then A_λ is invertible in ℓ_p and $\|A_\lambda^{-1}\|_p \le \left(\omega(1 - \sqrt[p]{\tau_p})\right)^{-1}$.

These results have some obvious consequences on the spectrum of the operator A and also on the type of hyperbolicity, see Remarks 1.10 and 1.11 of the thesis.

Finally, we applied these achievements to the ODE's $\dot{x} = A(t)x$ and $\ddot{x} = A(t)x$ with T-periodic $A(\cdot)$. The state space was set again on the infinite ℓ_p spaces. We do not present further details and the results here, interested reader should consult Subsection 1.3.3 of the thesis. The main point of the examinations was the combination of the well-known equivalent characterizations of the hyperbolic exponential dichotomy (c.f. [15]) with the above stated Theorem 6.

4 Prerequisites for results in Chapter 2

The main ODE, numerical schemes, the moving orthonormal system and some useful notations: Let us have $f \in C^3(\mathbb{R}^N)$, $N \in \mathbb{N} \setminus \{1\}$ such that $\varphi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is the global flow of $\dot{x} = f(x)$. For a numerical scheme $\psi : [0, h_0] \times \mathbb{R}^N \to \mathbb{R}^N, h_0 \in (0, 1)$ suppose for some $p \in \mathbb{N}$ that $\psi(h, x) = \varphi(h, x) + \Upsilon(h, x)h^{p+1}$. Assume again $\psi, \Upsilon \in C^3([0, h_0] \times \mathbb{R}^N, \mathbb{R}^N)$. Some technical reasons cause that we are forced to assume also $p \geq 2$ (see Remark 2.2 for more details).

Let $\gamma(s) := \varphi(s, \xi_0)$ be a 1-periodic solution for fixed $\xi_0 \in \mathbb{R}^N$. Then there is a system $\{e_i(s)\}_{i=1}^{N-1}$ of vectors in $\mathbb{R}^N$ for any $s \in \mathbb{R}$ such that

$$\begin{aligned} e_i \in C^3(\mathbb{R}, \mathbb{R}^N), \quad & e_i(s+1) = e_i(s), \\ \langle e_i(s), e_j(s) \rangle = \delta_{ij}, \quad & \langle e_i(s), f(\gamma(s)) \rangle = 0, \end{aligned}$$

where $i, j \in \{1, \cdots, N-1\}$, δ_{ij} is a Kronecker's delta and $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product. Introduce an $N \times (N-1)$ matrix $E(s) = [e_1, \cdots, e_{N-1}]$ (i-th column is e_i, $i = 1, \cdots, N-1$). Let us set also a tubular coordinate function $\xi(s, c) := \gamma(s) + E(s)c$ for $s \in \mathbb{R}, c \in \mathbb{R}^{N-1}$. For standard euclidian norm $|c|_2 := \sqrt{\langle c, c \rangle}$ note that $|E(s)c|_2 = |c|_2, c \in \mathbb{R}^{N-1}$. For $\delta > 0$ introduce the notation $B_{N-1}^{\delta} := \left\{ c \in \mathbb{R}^{N-1} : |c|_2 < \delta \right\}$. Using the implicit function theorem finite number of times we get that there is a $\delta_{\text{tr}} > 0$ such that $\xi : [0, 1) \times B_{N-1}^{\delta_{\text{tr}}} \to \mathbb{R}^N$ is a C^3–transformation, in other words $\xi|_{[0,1) \times B_{N-1}^{\delta_{\text{tr}}}}$ is a C^3–diffeomorphism between its domain and range (cf. the moving orthonormal system along γ in [32, Chapter VI.I., p. 214-219]) . For values $h \in [0, h_0]$, $s \in \mathbb{R}$, $c \in \mathbb{R}^{N-1}$, $\Delta \in [0, h_0]$, $X := (x^1, x^2, \cdots, x^{m-1}) \in \mathbb{R}^{N(m-1)}$, $x^i \in \mathbb{R}^N$, $m \in \mathbb{N}$, $m \geq 4$, define the following useful functions

$$\begin{aligned} F_m(h, s, c, X, \Delta) :=& (G_m(h, s, c, X), H_m(h, s, c, X, \Delta)), \\ G_m(h, s, c, X) :=& \big(\psi(h, \xi(s, c)) - x^1, \psi(h, x^1) - x^2, \psi(h, x^2) - x^3, \\ & \cdots, \psi(h, x^{m-2}) - x^{m-1}\big), \\ H_m(h, s, c, X, \Delta) :=& \left\langle \psi\left(\Delta, x^{m-1}\right) - \gamma(s), f(\gamma(s)) \right\rangle. \\ \bar{X}_m :=& \bar{X}_m(h, s, c) := \left(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^{m-1}\right), \\ \bar{x}^j :=& \bar{x}^j(h, s, c) := \varphi(jh, \xi(s, c)), \; j = 1, 2, \cdots, m-1. \end{aligned}$$

We mean by $|\cdot|$ the standard maximum norm $|v| := \max\{|v_i| : i = 1, \cdots, l\}$ for $v \in \mathbb{R}^l, l \in \mathbb{N}$. Notation $|\cdot|$ is used also for linear operators $A : \mathbb{R}^{l_1} \to \mathbb{R}^{l_2}$ defined as $|A| := \max_{v \in \mathbb{R}^{l_1}, |v|=1} |Av|$. An open ball in a Banach space X will be denoted as $B(x, \varrho) := \{y \in X : |y - x| < \varrho\}$ for any $x \in X$ and $\varrho > 0$.

The main tool of the chapter: It is the following specially designed lemma which follows the idea of the Newton–Kantorovich numerical method.

Lemma 7 (Lemma 2.1 of the thesis). *Let us have Banach spaces X, Y, Z and open nonempty sets $U \subset X, V \subset Y$. Let $\bar{y} : U \to V$ be any function such that $\overline{B(\bar{y}(x), \varrho)} \subset V$ for every $x \in U$ and for some $\varrho > 0$. Let us have a function $F \in C^r(U \times V, Z)$ for $r \geq 1$. Suppose that*

$D_yF(x,\bar{y}(x))^{-1} \in \mathcal{B}(Z,Y)$, $|F(x,\bar{y}(x))| \le \alpha$, $|D_yF(x,\bar{y}(x))^{-1}| \le \beta$ for every $x \in U$ and for some $\alpha, \beta > 0$. Let $|D_yF(x,y_1) - D_yF(x,y_2)| \le l|y_1 - y_2|$, $x \in U, y_1, y_2 \in \overline{B(\bar{y}(x),\varrho)}$ hold for some $l \ge 0$. For constants $\alpha, \beta, l, \varrho$ finally suppose $\beta l \varrho < 1$, $\alpha\beta < \varrho(1-\beta l\varrho)$. Then there is a unique function $\mathrm{y} : U \to V$ such that $|\mathrm{y}(x) - \bar{y}(x)|_Y \le \varrho$ and $F(x,\mathrm{y}(x)) = 0$ for all $x \in U$. Moreover $|\mathrm{y}(x) - \bar{y}(x)| < \varrho$ and $D_yF(x,\mathrm{y}(x))^{-1} \in \mathcal{B}(Z,Y)$ for all $x \in U$ with an estimation $|D_yF(x,\mathrm{y}(x))^{-1}| \le \frac{\beta}{1-\beta l\varrho}$. We get also $\mathrm{y} \in C^r(U,V)$ if we additionally assume the continuity of $\bar{y}$.

Global Poincaré map for the continuous DS: In the above described context the following assertion.

Lemma 8 (Lemma 2.2 in thesis, named as Poincaré's time return map)**.** *There is an $\varepsilon^\star \in (0,1/2)$ such that for every $\varepsilon \in (0,\varepsilon^\star]$ there is $\delta_{\mathrm{re}} = \delta_{\mathrm{re}}(\varepsilon) \in (0,\delta_{\mathrm{tr}}]$ and a C^3–function $\tau : \mathbb{R} \times B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon)} \to (1-\varepsilon, 1+\varepsilon)$ such that for $t \in (1-\varepsilon,1+\varepsilon), s \in \mathbb{R}$ and $c \in B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon)}$ we have $z(t,s,c) = 0$ for $z(t,s,c) := \langle \varphi(t,\xi(s,c)) - \gamma(s), f(\gamma(s))\rangle$ if and only if $t = \tau(s,c)$. In addition $\tau(s+1,\cdot) = \tau(s,\cdot), s \in \mathbb{R}$.*

Now the usual Poincaré map is defined as $\mathcal{P}(s,c) := \varphi(\tau(s,c),\xi(s,c))$. Further for admissible values of (h,s,c) let us introduce $\bar{\Delta}_m := \bar{\Delta}_m(h,s,c) := \tau(s,c) - (m-1)h$.

5 New results in Chapter 2

We will not state the precise form of the key theorem about the numerical Poincaré map, because it needs quite a lot of preparatory technicalities. In order to give some insights we assert below its weaker, more indefinite variant.

Theorem 9 (weakened form of Theorem 2.3 of the thesis)**.** *For $C > 0$ large enough, $|h - 1/m|, |c|$ small enough, any $s \in \mathbb{R}$ and for any m large enough there exists a unique pair $(X_m, \Delta_m) = (X_m(h,s,c), \Delta_m(h,s,c))$ such that*

$$F(X_m,\Delta_m) = F_m(h,s,c,X_m(h,s,c),\Delta_m(h,s,c)) = 0$$

and $|X_m - \bar{X}_m| < C/m^p$, $|\Delta_m - \bar{\Delta}_m| < C/m^p$. Moreover X_m, Δ_m are C^3–smooth in their arguments and 1-periodic in s.

The proof of Theorem 9 is nothing else then an application of Lemma 7 in a suitable framework. A lot of work was done in the thesis in order to specify the exact sufficient merits of quantities $C, |h-1/m|, |c|, m$. The whole Chapter 2 can be characterized as an "expedition" among various constants. Having at hand the theorem above we can define a natural approximation of $\mathcal{P}$ as

$$\mathcal{P}_m(h,s,c) := \psi\left(\Delta_m(h,s,c), x_m^{m-1}(h,s,c)\right)$$

which we call the numerical (or dicretized) Poincaré map.

In Section 2.3 we dealt with various bounds for the term $|D_v\,[\mathcal{P}-\mathcal{P}_m]|$, where $v\in\{h,s,c\}$ and D_v denotes the partial differentiation with respect to variable v. After a lengthy computational part we obtained $|D_h[\mathcal{P}(s,c)-\mathcal{P}_m(h,s,c)]|\le\kappa_h/m^{p-1}$, and $|D_v[\mathcal{P}(s,c)-\mathcal{P}_m(h,s,c)]|\le\kappa_v/m^p$, for $v\in\{s,c\}$, m large enough (κ_v for $v\in\{h,s,c\}$ were properly described constants). For details see Theorem 2.5 and Remark 2.3 of the thesis. Let us mention that the main idea was to improve Lemma 7. Namely, using the notations of Lemma 7 an estimation for $|\mathrm{y}'(x)-\bar{y}'(x)|$ was given under additional assumptions. After these tasks we gave some results regarding to the second derivatives (cf. the end, p. 69 - 73, of Section 2.3 of the thesis).

In Section 2.4 of the thesis we showed an application of the preceding theory. We proved there the slightly stronger version of the following theorem.

Theorem 10 (weakened form of Theorem 2.7 of the thesis)**.** *Suppose the non-degeneracy of γ, that is: 1 is a simple eigenvalue of $\varphi'_x(1,\xi_0)$. Then for every m large enough there is a function $(h,s)\to\zeta_m(h,s)$ such that $\mathcal{P}_m(h,s,\zeta_m(h,s))=\xi(s,\zeta_m(h,s))$ is valid, where $s\in\mathbb{R}$ and $|h-1/m|$ is small enough. In addition ζ_m is C^3-smooth and 1-periodic in s.*

Moreover the uniqueness of ζ_m was also shown in an adequate sense. The proof was again an application of Lemma 7. Let us mention that the curve $s\in\mathbb{R}\to\xi(s,\zeta_m(h,s))$ for fixed h is invariant under $\mathcal{P}_m(h,\xi^{-1}(\cdot))$ (see Remark 2.4 of the thesis).

In the end of Chapter 2 we stated and proved (with our methods) an already known result about the curve of the m-periodic points for the discrete dynamics (cf. [21] and Theorem 2.8 of the thesis). The whole thesis was finished by a contribution on the spectrum of the established curves (see Remark 2.5 of the thesis).

6 Summary

In the first part of the thesis we have considered a differential inclusion $\dot{x} \in A(t)x + f(t,x) + g(t,x,X_1)$ in a Banach space X with a general exponential dichotomy, where X_1 is the closed unit ball of X. We assumed that the right-hand side is strongly measurable in the time variable and Lipschitz continuous in the others. We proved the existence and uniqueness of quasibounded solutions corresponding to suitable selectors. Analogues of stable and unstable manifolds were introduced and a graph characterization was given. We showed some deeper properties of these multivalued manifolds concerning their hierarchy and independence on a special parameter. These kinds of inclusions model among others the effect of roundoff error in the numerical analysis of dynamical systems. The first chapter was closed with various sufficient criteria for hyperbolic exponential dichotomy.

The next chapter was devoted to the analytical study of the relationship between the Poincaré map and its one step discretization. Error estimates were established depending basically on the right-hand side function of the investigated ODE and the given numerical scheme. Our basic tool in this chapter was a parametric version of the Newton–Kantorovich method. Applying these results, in the neighborhood of a non-degenerate periodic solution a new type of step-dependent, closed curve was detected for the discrete dynamics. The discretized Poincaré map is a preparatory stage for further investigation of bifurcations of discrete dynamics near periodic solutions.

Key words: multivalued analysis, integral manifolds, exponential dichotomy, hierarchy, Poincaré map, discrete dynamics

Publications related to the thesis:

M. FEČKAN & S. KELEMEN: *Multivalued Integral Manifolds in Banach Spaces*, Communications in Mathematical Analysis, Vol. 10, No. 2 (2011), p. 97-117,

M. FEČKAN & S. KELEMEN: *Discretization of Poincaré Map*, submitted to Electronic Journal of Qualitative Theory of Differential Equations.

Other publication:

S. KELEMEN & P. QUITTNER: *Boundedness and a priori estimates of solutions to elliptic systems with Dirichlet-Neumann boundary conditions*, Communications on Pure and Applied Analysis, Vol. 9, No. 3 (2010), p. 731-740.

– Cited in: I. KOSÍROVÁ: *Regularity and a priori estimates of solutions for semilinear elliptic systems*, Acta Mathematica Universitatis Comenianae, Vol. 79, No. 2, 2010, p. 231-244.

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