COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

Multivalued Integral Manifolds in Banach Spaces and the Numerical Poincaré Map

Dissertation thesis

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Mgr. Sándor Kelemen

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External Educational Institute:	Mathematical Institute, Slovak Academy of Sciences
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I am grateful to God for that I live in a place where undisturbed work with what I like is possible. I would like to thank my supervisor prof. RNDr. Michal Fečkan DrSc. for that I could count on Him in every situation. Without His helpfulness and unique mathematical knowledge this work would never have been created. I am also deeply thankful to my beloved wife, my family and friends for their encouragements and all kinds of support during my scientific work.

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Abstract

In the first part of the thesis we consider a differential inclusion

$$\dot{x} \in A(t)x + f(t,x) + g(t,x,X_1)$$

in a Banach space X with a general exponential dichotomy, where X_1 is the closed unit ball of X. The right-hand side is strongly measurable in the time variable and Lipschitz continuous in the others. We prove the existence and uniqueness of quasibunded solutions corresponding to suitable selectors. Analogues of stable and unstable manifolds are introduced and a graph characterization is given. We show some deeper properties of these multivalued manifolds concerning their hierarchy and independence on a special parameter. These kinds of inclusions model among others the effect of roundoff error in the numerical analysis of dynamical systems. The first chapter is closed with various sufficient criteria for exponential dichotomy.

The next chapter is devoted to the analytical study of the relationship between the Poincaré map and its one step discretization. Error estimates are established depending basically on the right-hand side function of the investigated ODE and the given numerical scheme. Our basic tool in this chapter is a parametric version of the Newton–Kantorovich method. Applying these results, in the neighborhood of a non-degenerate periodic solution a new type of step-dependent, closed curve is detected for the discrete dynamics. The discretized Poincaré map is also a preparatory stage for further investigation of bifurcations of discrete dynamics near periodic solutions.

Key words: multivalued analysis, integral manifolds, exponential dichotomy, hierarchy, Poincaré map, discrete dynamics

Abstrakt

V prvej časti dizertačnej práce uvažujeme diferenciálnu inklúziu

$$\dot{x} \in A(t)x + f(t, x) + g(t, x, X_1)$$

v Banachovom priestore X so všeobecnou exponenciálnou dichotómiou, kde X_1 je jednotková uzavretá guľa v X. Pravá strana rovnice je silne merateľná v časovej premennej a lipschitzovsky spojitá v ostatných. V práci je dokázaná existencia a jednoznačnosť kváziohraničených riešení zodpovedajúcich vhodným selektorom. Je uvedená analógia stabilných a nestabilných variet a je dokázaná ich grafová charakteristika. Ďalej sú ukázané niektoré hlbšie vlastnosti týchto viacznačných variet, ich hierarchia a nezávislosť na špeciálnom parametri. Diferenciálne inklúzie týchto typov modelujú okrem iného efekt zaokrúhlovacej chyby numerickej analýzy dynamických systémov. Na konci prvej časti sú prezentované rôzne nutné kritériá exponenciálnej dichotómie.

Následná kapitola je venovaná analytickému skúmaniu vzťahu medzi Poincarého zobrazením a jeho jednokrokovou diskretizáciou. Chyby odhadov sú dané v závislosti od pravej strany skúmanej ODR a použitej numerickej schémy. Základným nástrojom tejto kapitoly bola parametrická verzia Newtonovej–Kantorovičovej metódy. Aplikáciou týchto výsledkov, v okolí nedegenerovaného periodického riešenia bola objavená nová, krokovo závislá, uzavretá krivka pre diskrétnu dynamiku. Diskretizované Poincarého zobrazenie je základom ďalšieho skúmania bifurkácií diskretizovanej dynamiky v okolí periodických riešení.

Kľúčové slová: viacznačná analýza, integrálne variety, exponenciálna dichotómia, hierarchia, Poincarého zobrazenie, diskrétna dynamika

Preface

The present thesis is the result of my four year long doctoral studies at the Mathematical institute of Slovak Academy of Sciences. The thesis is an adaptation and extension of two papers written under the leadership of my supervisor (c.f. [25, 26] – the first one was already published, the second one has been submitted and we have received an affirmative answer recently).

Our objects of studies are the numerical notions in dynamical systems, namely the inflated dynamics and the numerical Poincaré map. We have chosen this topic firstly because we hope that the numerical aspects of the "pure" theory of continuous dynamical systems (DS) keep our results closer to some real-word applications. Secondly we have been highly motivated by the typical question: What properties of DS persist under the various numerical procedures applied to the continuous DS? The goal of our thesis is at first to give an adequate description for the analogous of the stable and unstable manifolds in a multivalued setting for differential inclusions possessing exponential dichotomy on $\mathbb{R}$. Second, to introduce a numerical Poincaré map with a rigorous analytical approach.

We hope that our investigations bring some new insights to the perturbations of continuous DS and also that it might lead to some improvements in the future regarding the general understanding of numerical procedures.

Our thesis is written on the level of mathematics which is accessible to the reader having a basic knowledge of mathematical analysis. Rarely occurring advanced topics are everywhere properly explained.

Bratislava, 29$^{\text{th}}$ of April, 2013
Sándor Kelemen

Contents

Introduction

As we already mentioned in the Preface, this thesis is a contribution to the present knowledge of the dynamics of numerical procedures applied to continuous dynamical system (DS). The emphasis is made on two topics discussed in two independent chapters. Now, we would like to give some brief insights into these topics.

It is well-known that invariant manifolds like stable and unstable ones play an important role in understanding saddle dynamics for smooth nonlinear DS [3,36,38, 55–57]. To the best of our knowledge, instead, there are only few papers dealing with saddle dynamics for non-smooth or even *multivalued* DS. The simplest example of multivalued DS is the so-called *inflated dynamics*, which was introduced in [39] and it was used in a fairly large number of papers since then (for details, see [30]). Chapter 1 is a continuation of [12], and we refer the reader for more results and a discussion on multivalued hyperbolic dynamics to that paper. Like in [12], our multivalued DS takes a special form of a *parametrized*, i.e. *controlled* form with Lipschitzian nonlinearities/multifunctions. In view of a parameterization result by Ornelas [46], this is not a loss of generality in finite dimensional cases and with convex valued Lipschitzian multifunctions. However, in the general case such a parameterization does not exist, see the Appendix of [12] for a short discussion about the parameterization problem for multifunctions. Multivalued differential equations are closely related to numerical procedures and they serve a good model in order to include the step-to-step error terms to the exact model. Such an errors always occur in numerical realizations.

In the first chapter we consider parameterized Lipschitzian and Carathéodorian semi-linear differential inclusions in Banach spaces with exponentially dichotomous linear parts. Under additional assumptions, we prove the existence and uniqueness of quasibounded solutions. Then the analogy of the stable and unstable sets corresponding to these quasibounded solutions are defined and it turns out that they are the graphs of suitable multifunctions. We also introduce and study solutions corresponding to more general weighted selector spaces. We discuss hierarchy like in [3]. Chapter 1 is concluded with presenting some criteria on the existence of hyperbolic exponential dichotomy on $\mathbb{R}$. These sufficient conditions are derived for constant matrices on a finite dimensional $\mathbb{C}^n$, for a class of infinite matrices on complex ℓ_p spaces and finally for some non-autonomous periodic ODE's also on ℓ_p.

After that, Chapter 2 is devoted to the precise analytical derivation of the numerical/discretized Poincaré map of an ordinary differential equation possessing a periodic orbit. We have been motivated by papers [33, 64], where numerical tools are used for computing the Poincaré map. There is a vast amount of materials concerning dynamics of numerical approximations of ODE, see for instance [23, 27, 28, 31, 59, 60]. This chapter of the thesis gives a contribution to this direction.

The continuous Poincaré map $\mathcal{P}$ for the smooth ODE with a 1-periodic orbit γ is a well-understood topic and is contained in almost every textbook on continuous dynamical systems (e.g. [45]). To define the numerical Poincaré map $\mathcal{P}_m$ for a discretized dynamical system obtained from the one-step discretization procedure we have chosen a method originating in [33]. Here m designate the first natural number of the whole-step realizations of the discretization scheme such that the next, $m+1$-th realization, exceed the Poincaré section. Our goal is to give a precise analytical meaning of $\mathcal{P}_m$ and to establish error bounds for the difference $|\mathcal{P} - \mathcal{P}_m|$ and its various differentials. Accuracies are given in a form $\frac{C}{m^q}$ for every m large enough. Here the constant C essentially depends on the right-hand side of the ODE, on the particular numerical scheme and q is usually a simple linear function of the order p of the scheme. Achieved results, as we have anticipated, correspond to [28] where the author examined the C^j–closeness, $j \geq 0$, between the flow and its numerical approximation. Our approach uses the method of a moving orthonormal system (introduced rigorously in [32] and then used successfully in [6, 8, 59]) and the Newton–Kantorovich type theorem (cf. [37, 47, 67]).

In the end of Chapter 2 we apply the previously established properties of $\mathcal{P}_m$. It is a slight completion of the paper [21] where two closed curves were found in a neighborhood of the orbit γ. The first one is the set of m-periodic points x of the discretization scheme, where the step h of the scheme depends on x and is close enough to $1/m$. The existence and uniqueness of this curve was shown under the *nondegeneracy condition* of γ. The second curve, the maximal invariant set of the scheme in a neighborhood of γ, was derived basically under the hyperbolicity of γ, for any sufficiently small step (this is a historically well-known topic, it was treated for example in [6, 8, 22, 53]). With our setting we show in Section 2.4 that the nondegeneracy of γ in a small surrounding neighborhood leads us to the third interesting curve. Namely the set of those points which are invariant under the action of $\mathcal{P}_m$. We establish the existence and uniqueness for any m large enough and any h close enough to $1/m$. We also give a short remark about the spectral property of this curve.

Chapter 1

Multivalued Integral Manifolds

1.1 Preliminaries

1.1.1 Measure Theory

Throughout the whole chapter except Section 1.3 we suppose that X is a real Banach space (a complete normed space over the field $\mathbb{R}$). We say that an interval $I \subset \mathbb{R}$ of arbitrary type is *positive* if $|I| > 0$ for its (Lebesgue) measure (the case $+\infty$ is also involved). For this subsection assume that $I \subset \mathbb{R}$ is a nonempty interval.

The function $f : I \to X$ is *strongly measurable*[1] (abbreviation *s. m.*) if the range $f(I)$ is separable and f is measurable (f is measurable if the pre-image $f^{-1}(B)$ is a Borel set for all Borel sets $B \subset X$). Further f is *simple* if it has only finitely many values and is strongly measurable. A fundamental fact about s. m. functions is the following lemma.

Lemma 1.1 (The approximation property of s. m. functions)**.** *Suppose that we have a s. m. function $f : I \to X$. Then there is a sequence $\{f_n\}_{n=1}^{\infty}$ of simple functions which converges pointwise to f and satisfy the estimation $|f_n(t)| \leq |f(t)|$ for* all $t \in I$. *Moreover every pointwise limit of a sequence of s. m. functions is again strongly measurable, in other words, the set of s. m. functions is closed under the formation of pointwise limits.*

Proof. See [11, Appendix E]. □

A function $f : I \times X \to X$ has a *Carathéodory property* if at one hand $f(t, \cdot) : X \to X$ is continuous for all fixed $t \in I$ and on the other hand $f(\cdot, x) : I \to X$ is s. m. for

[1] There are slightly different approaches of this notion, we follow [3, 11].

all fixed $x \in X$. We denote the set of these function by $\mathcal{CAR}(I, X)$. The following consequence of Lemma 1.1 will be useful for us.

Lemma 1.2. *Suppose that $f \in \mathcal{CAR}(I, X)$ and $\mu : I \to X$ is a s. m. function, then the partially composed mapping $g : I \to X$ defined as $g(t) := f(t, \mu(t))$ is also a s. m. function.*

Proof. As the proof of [3, Lemma 2.2] where continuous μ was investigated. □

We suppose that the reader has been acquainted with the theory of Lebesgue integrals. The brief definition of Bochner integrals using Lebesgue integrals is the following one: a s. m. function $f : I \to X$ is *Bochner integrable* (or simply *integrable*) if the norm function $|f| : I \to \mathbb{R}$ defined as $|f|(t) := |f(t)|$ is Lebesgue integrable. The function f is called *locally integrable* if it is s. m. on I and integrable over compact subintervals of I.

For an integrable simple function $f = \sum_{j=1}^{k} \alpha_j \chi_{I_j}$ where $\alpha_j \in \mathbb{R}, I_j \subset I$ are measurable and χ_{I_j} is the characteristic function of the set I_j we define the Bochner integral as

$$\int_i f \mathrm{d}t := \sum_{j=1}^{k} a_j |I_j|.$$

For an arbitrary integrable function from Lemma 1.1 we get simple integrable functions f_n such that $f = \lim_{n \to \infty} f_n$, $|f_n(t)| \leq |f(t)|$. Then the well-known Lebesgue's Dominated Convergence Theorem for real-valued functions implies the well-definiteness of

$$\int_I f \mathrm{d}t := \lim_{n \to \infty} \int_I f_n \mathrm{d}t.$$

1.1.2 The Uniform Contraction Principle

One of the most often used tools in the theory of differential equations is the Uniform Contraction Principle. We do not formulate it in the most general version, but only the case what we use. In general $| \cdot |_X$ will denote the norm in a Banach space X, however in most of the cases there are no arising confusions so we use simply a notation $| \cdot |$.

Theorem 1.3. *Assume that X is a Banach space, $\mathcal{M}$ is a nonempty metric space with a metric d and $f : X \times \mathcal{M} \to X$ is a uniform contraction that is there exist $\alpha \in [0, 1)$ such* *that*

$$|f(x_1,y)-f(x_2,y)| \le \alpha |x_1-x_2|, \qquad x_1,x_2 \in X, y \in \mathcal{M}.$$

Then for all $y \in \mathcal{M}$ there exists a unique fixed point $\mathbf{x} = \mathbf{x}(y)$ of the function $f(\cdot, y) : X \to X$. In addition $\mathbf{x} : \mathcal{M} \to X$ is continuous under the continuity assumption of $f(x,\cdot) : \mathcal{M} \to X$ for every $x \in X$. The uniform Lipschitz property

$$|f(x,y_1)-f(x,y_2)| \le Ld(y_1,y_2), \qquad x \in X, y_1,y_2 \in \mathcal{M}$$

implies

$$|\mathbf{x}(y_1)-\mathbf{x}(y_2)| \le \frac{L}{1-\alpha} d(y_1,y_2), \qquad x \in X, y_1,y_2 \in \mathcal{M}.$$

Proof. The existence and the uniqueness of $\mathbf{x}(y)$ is a straightforward application of Banach's Fixed Point Theorem (cf. [17, Theorem 7.1, p. 39]). The continuity and the Lipschitz property follows from the identity

$$|\mathbf{x}(y_1)-\mathbf{x}(y_2)| \le \frac{1}{1-\alpha} \left| f(\mathbf{x}(y_1),y_1) - f(\mathbf{x}(y_1),y_2) \right|$$

which is a consequence of

$$\begin{aligned} |\mathbf{x}(y_1)-\mathbf{x}(y_2)| &= |f(\mathbf{x}(y_1),y_1) - f(\mathbf{x}(y_2),y_2)| \\ &\le |f(\mathbf{x}(y_1),y_1) - f(\mathbf{x}(y_1),y_2)| + |f(\mathbf{x}(y_1),y_2) - f(\mathbf{x}(y_2),y_2)| \\ &\le |f(\mathbf{x}(y_1),y_1) - f(\mathbf{x}(y_1),y_2)| + \alpha \left|\mathbf{x}(y_1)-\mathbf{x}(y_2)\right|. \end{aligned}$$

□

1.1.3 Solution Concepts, Inflated Differential Equations

Suppose for this subsection that I is a positive interval and $\mathcal{M}$ is a topological space. Our subject of investigation will be the ordinary differential equations (ODE)

$$\dot{x} = f(t,x,y) \tag{1.1}$$

for parameters $y \in \mathcal{M}$ and the *inflated differential equation* (IDE)

$$\dot{x} \in F(t, x, X_1) \tag{1.2}$$

where $X_1 = \{y \in X : |x| \le 1\}$ and $F(t, x, X_1) = \{F(t, x, u) : u \in X_1\}$ for an arbitrary function $F : I \times X \times X_1 \to X$. Here and later on Y_1 designate the closed unit ball for any Banach space Y.

Definition 1.4. Assume that J is a positive subinterval of I and $f : I \times X \times \mathcal{M} \to X$ satisfies $f(\cdot, \cdot, y) \in \mathcal{CAR}(I, X)$ for all $y \in \mathcal{M}$. A continuous function $\lambda : J \to X$ is a *solution* of the ODE (1.1) at the parameter value $y \in \mathcal{M}$ if the function $f(\cdot, \lambda(\cdot), y) : J \to X$ is locally integrable and

$$\lambda(t) - \lambda(s) = \int_s^t f(\tau, \lambda(\tau), y) \mathrm{d}\tau$$

holds for all $s, t \in J$. In addition we say that λ satisfies the *initial condition* $x(t_0) = x_0$ for some fixed values $t_0 \in I, x_0 \in X$ if $t_0 \in J$ and $\lambda(t_0) = x_0$.

Remark 1.1. A function $f : J \to X$ is said to be *absolutely continuous* (abbreviated a. c.) if there is an integrable function $g : J \to X$ such that $f(t) = f(s) + \int_s^t g(\tau)\mathrm{d}\tau$ for all $s, t \in J$. *Locally a. c.* means a. c. on every compact subinterval. Note that in the context of Definition 1.4 a solution λ of (1.1) is automatically locally absolutely continuous on J. The fundamental properties of a. c. functions implies that λ is differentiable a. e. in J and (1.1) is valid a. e. in J (a. e. is used for “almost everywhere”, it means that the measure of the set where the given property does not hold is zero). The given notion in Definition 1.4 is called sometimes absolutely continuous or Carethedéodory or generalized solutions (c.f. [40, Chapter 18]).

Let us recall the following fundamental theorem about existence, uniqueness and continuous dependence of these solutions.

Theorem 1.5 (See Theorem 2.4 in [3]). *Suppose that*

$$f : I \times X \times \mathcal{M} \to X, \qquad f(\cdot, \cdot, y) \in \mathcal{CAR}(I, X) \text{ for all } y \in M.$$

Assume also with locally integrable functions $l, l_0 : I \to \mathbb{R}_0^+$ the following conditions

$$\begin{aligned} |f(t, x_1, y) - f(t, x_2, y)| &\leq l(t)|x_1 - x_2|, \\ |f(t, 0, y)| &\leq l_0(t), \end{aligned}$$

for almost all $t \in I$, for all $x_1, x_2 \in X$ and $y \in \mathcal{M}$. Finally, suppose that $f(t, x, \cdot) : \mathcal{M} \to X$ is continuous for all $(t, x) \in I \times X$. Then the initial value problem

$$\dot{x} = f(t, x, y), \qquad x(t_0) = x_0$$

has a unique solution $\lambda(\cdot; t_0, x_0, y) : I \to X$ for all $(t_0, x_0, y) \in I \times X \times \mathcal{M}$. In addition the so-defined mapping $\lambda : I \times I \times X \times \mathcal{M} \to X$ is continuous.

Let J be a positive interval. We introduce a function space

$$\mathsf{H}(J) := \{h : J \to X \ : \ h \text{ is s. m. and } |h|_{J,\infty} < \infty\} \tag{1.3}$$

where $|h|_{J,\infty} = \sup_{t \in J} |h(t)|$. It is easy to see that $\mathsf{H}(J)$ endowed with the norm $|\cdot|_{J,\infty}$ is a Banach space. For simplicity we introduce also

$$\mathsf{H} := \mathsf{H}(\mathbb{R}), \quad |\cdot|_\infty := |\cdot|_{\mathbb{R},\infty}, \qquad \mathsf{H}_\tau^\pm := \mathsf{H}(\mathbb{R}_\tau^\pm), \quad |\cdot|_\tau^\pm := |\cdot|_{\mathbb{R}_\tau^\pm,\infty}.$$

where $\mathbb{R}_\tau^+ := [\tau, \infty)$ and $\mathbb{R}_\tau^- := (-\infty, \tau]$ for $\tau \in \mathbb{R}$. We will refer to the elements of $\mathsf{H}(J)$ as *selectors*.

Definition 1.6. Assume that $J \subset I$ for positive intervals J, I. We say that the continuous function $\lambda : J \to X$ is a *solution* of the IDE (1.2) corresponding to the *selector* $h \in \mathsf{H}(J)_1$ if λ is a solution of $\dot{x} = F(t, x, h(t))$ (in the sense of Definition 1.4). In addition we say that λ satisfies the *initial condition* $x(t_0) = x_0$ for $t_0 \in I, x_0 \in X$ if we have $t_0 \in J$ and $\lambda(t_0) = x_0$.

Remark 1.2. Consider a multivalued mapping $\tilde{F}$ from $I \times X$ to the set of all subsets of X (usual notation is $F : I \times X \rightsquigarrow X$). Then we should investigate a more general *differential inclusion* $\dot{x} \in \tilde{F}(t, x)$. Here a solution would be an a. c. function $\lambda : J \to X$ for which the relation $\dot{x}(t) \in \tilde{F}(t, x(t))$ holds for a. e. $t \in J$. However, following papers [12, 13] we cover only those differential inclusions where $\tilde{F}$ has a suitable single valued *parametrization* $F : I \times X \times X_1 \to X$, that is $\tilde{F}(t,x) = F(t,x,X_1)$ (about the question of the existence of such a parametrization see the thorough studies in [1, 2, 18, 58, 63]; for example in the case $X = \mathbb{R}^n$ with a convex valued $\tilde{F}$ which is *continuous* in x (we do not present the definition of the continuity of multivalued mappings – interested reader should consult [2]) a parametrization is always possible – c.f. [46]). Moreover in a case when $F(\cdot, x, u) : I \to X$ is measurable and $F(t, \cdot, \cdot) : X \times X_1 \to X$ is continuous we can justify our Definition 1.6 because then the set of those locally a. c. functions which satisfies the relation $\dot{x} \in F(t, x, X_1)$ a. e. coincides with the set of solutions in a sense of our Definition 1.6 (c.f. [34, Theorem 7.2]). The indubitable advantage of Definition 1.6 is that the final object of consideration is a differential *equation* with a right-hand side parameterized over the function space $\mathsf{H}(J)_1$.

Theorem 1.7. *Assume that I is a positive interval and the right-hand side function F : $I \times X \times X_1 \to X$ satisfies the following three requirements:*

(i) $F(\cdot, x, u) : I \to X$ *is s. m. for all fixed* $(x, u) \in X \times X_1$,

(ii) $F(t, \cdot, \cdot) : X \times X_1 \to X$ *is continuous for all* $t \in I$,

(iii) there exists a locally integrable functions $l_0, l_1, l_2 : I \to [0, \infty)$ *such that*

$$\begin{aligned} |F(t, x_1, u_1) - F(t, x_2, u_2)| &\leq l_1(t)|x_1 - x_2| + l_2(t)|u_1 - u_2|, \\ |F(t, 0, 0)| &\leq l_0(t), \end{aligned}$$

for a. e. $t \in I$ *and for all* $x_1, x_2 \in X, u_1, u_2 \in X_1$.

Under these conditions for every triple $(t_0, x_0, h) \in I \times X \times \mathsf{H}(I)_1$ *there exists a unique solution* $\lambda(\cdot) = \lambda(\cdot; t_0, x_0, h) : I \to X$ *of the initial value problem*

$$\dot{x} = F(t, x, h(t)), \qquad x(t_0) = x_0.$$

In other words, for every $(t_0, x_0) \in I \times X$ *the initial value problem*

$$\dot{x} \in F(t, x, X_1), \qquad x(t_0) = x_0$$

has a unique solution $\lambda(\cdot) = \lambda(\cdot; t_0, x_0, h)$ corresponding to the selector $h \in \mathsf{H}(I)_1$. In addition the mapping $\lambda : I \times I \times X \times \mathsf{H}(I)_1 \to X$ is continuous.

Proof. The proof follows the lines of [3, Theorem 2.4]. We discuss only the main idea. Without loss of generality we may restrict our attention to the case $I = [a, b]$ for arbitrary fixed $a < b, a, b \in \mathbb{R}$ (the simple reason is that every positive interval I can be written in the form $I = \bigcup_{j \in \mathbb{N}} [a_j, b_j]$ where $a_{j+1} \leq a_j < b_j \leq b_{j+1}$ for $j \in \mathbb{N}$ and $a_j, b_j \in \mathbb{R}$).

Denote by $C(I, X)$ the Banach space of continuous functions $x : I \to X$ with a norm $|\cdot|_\infty$. Define an operator $\mathcal{T}$ for $x \in C(I, X)$ and $t_0, t \in I, x_0 \in X, h \in \mathsf{H}(I)_1$ as

$$\mathcal{T}(x; t_0, x_0, h)(t) := x_0 + \int_{t_0}^{t} F(s, x(s), h(s))\mathrm{d}s.$$

It turns out that $\mathcal{T} : C(I, X) \times I \times X \times \mathsf{H}(I)_1 \to C(I, X)$ is well-defined and continuous. Moreover for $n \in \mathbb{N}$ sufficiently large the iterated mapping $\mathcal{T}^n$ (defined as $\mathcal{T}^k(x; t_0, x_0, h) := \mathcal{T}(\mathcal{T}^{k-1}(x; t_0, x_0, h); t_0, x_0, h)$ for $k \geq 2$ and $\mathcal{T}^1 := \mathcal{T}$) is a uniform contraction at the first variable. Applying Theorem 1.3 we get a unique fixed point x of $\mathcal{T}^n$ which depends continuously on the "parameters" t_0, x_0, h. Now $\mathcal{T}(\mathbf{x}) = \mathcal{T}(T^n(\mathbf{x})) = \mathcal{T}^n(\mathcal{T}(\mathbf{x}))$ (t_0, x_0, h omitted) therefore using the uniqueness of x we have $\mathcal{T}(\mathbf{x}) = \mathbf{x}$ and the proof is complete. □

1.1.4 Non-autonomous Systems

For a Banach space X we introduce $\mathcal{B}(X)$ the space of bounded and linear operators $L : X \to X$. The norm in $\mathcal{B}(X)$ is given as

$$|L|_{\mathcal{B}(X)} := \sup_{|x|_X = 1, x \in X} |Lx|_X, \qquad L \in \mathcal{B}(X).$$

Let us have a positive interval I and an arbitrary function $A : I \to \mathcal{B}(X)$.

Definition 1.8. We say that $\Phi : I \times I \to \mathcal{B}(X)$ is the *evolution operator* of the non-autonomous homogeneous linear differential equation

$$\dot{x} = A(t)x, \qquad x \in X$$

if $\Phi(\cdot, s) : I \to \mathcal{B}(X)$ is a solution of the operator valued ($Y(\cdot) \in \mathcal{B}(X)$) initial value problem

$$\dot{Y} = A(t)Y, \qquad Y(s) = \mathbb{I}_X.$$

Sometimes it is also convenient to use the solution of the operator differential equation $\dot{Y} = A(t)Y$ on I with the initial condition $Y(0) = \mathbb{I}_X$. It is denoted by $\widetilde{\Phi}$ and it is a function from I to $\mathcal{B}(X)$. The obvious relations between Φ and $\widetilde{\Phi}$ are $\widetilde{\Phi}(t) = \Phi(t, 0)$ and $\Phi(t, s) = \widetilde{\Phi}(t) \circ \left(\widetilde{\Phi}(s)\right)^{-1}$.

Consequence 1.9 (of Theorem 1.5)**.** *Suppose that $A : I \to \mathcal{B}(X)$ is locally integrable (see Subsection 1.1.1) then Theorem 1.5 ensures that there is a unique evolution operator $\Phi : I \times I \to \mathcal{B}(X)$ of $\dot{x} = A(t)x$. In addition Φ is continuous.*

Proof. Apply Theorem 1.5 in a framework $f : I \times \mathcal{B}(X) \to \mathcal{B}(X)$, $f(t, Y) = A(t)Y$ and $l(t) = |A(t)|$, $l_0 = 0$. Then set $\Phi(t, s) := \lambda(t, s, \mathbb{I}_X)$. □

From now on we always suppose at least that A is locally integrable. We collect some important properties of Φ to the following lemma.

Lemma 1.10. *For all $t, s \in I$ we have $\Phi(t, s) \in GL(X)$ which is the group of invertible operators in $\mathcal{B}(X)$. Moreover $\Phi : I \times I \to \mathcal{B}(X)$ is continuous, $|\Phi(t, s)|_{\mathcal{B}(X)} \leq e^{|\int_s^t |A(\tau)| d\tau|}$ for all $s, t \in I$ and the following* cocycle property *is valid*

$$\Phi(t_3, t_2) \circ \Phi(t_2, t_1) = \Phi(t_3, t_1), \quad t_1, t_2, t_3 \in I.$$

In addition for any locally integrable function $f : I \to X$ there exists a unique solution of the inhomogeneous linear differential equation $\dot{x} = A(t)x + f(t)$ with initial condition $x(t_0) = x_0 \in X, t_0 \in I$. Moreover this solution is given by an explicit formula the so-called variation of constants *formula*

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s) f(s) \mathrm{d}s. \tag{1.4}$$

Proof. See Lemma 2.9 and Theorem 2.10 in [3]. □

Definition 1.11. We say that the equation $\dot{x} = A(t)x$ possesses an *exponential dichotomy* on I if there are constants $K \geq 1$, $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}$ and a *projection* $P \in \mathcal{B}(X)$

(projection means that $P^2 = P$) such that

$$\left.\begin{aligned} \left|\widetilde{\Phi}(t)\circ P^{+}\circ\big(\widetilde{\Phi}(s)\big)^{-1}\right|_{\mathcal{B}(X)} &\le K\,\mathrm{e}^{\alpha(t-s)}, \quad t\ge s,\ t,s\in I \\ \left|\widetilde{\Phi}(t)\circ P^{-}\circ\big(\widetilde{\Phi}(s)\big)^{-1}\right|_{\mathcal{B}(X)} &\le K\,\mathrm{e}^{\beta(t-s)}, \quad t\le s,\ t,s\in I \end{aligned}\right\} \tag{1.5}$$

where $P^+ := P$ and $P^- := \mathbb{I}_X - P^+$. We denote by $\mathcal{E}_{\alpha,\beta}(I)$ the set of all locally integrable $A : \mathbb{R} \to L(X)$ for which $\dot{x} = A(t)x$ possesses an exponential dichotomy on I.

Furthermore we introduce notations

$$P^{\pm}(t) := \widetilde{\Phi}(t)\circ P^{\pm}\circ\big(\widetilde{\Phi}(t)\big)^{-1}, \quad \mathbb{P}^{\pm}_t := P^{\pm}(t)(X).$$

Note that $P^{\pm}(t)$ are projections and $P^+(t) + P^-(t) = I$. It is easy to establish the quasi-commutation $P^{\pm}(t)\circ\Phi(t,s) = \Phi(t,s)\circ P^{\pm}(s)$ for any $s,t\in I$. Now (1.5) could be equivalently formulated using the co-cycle property of Φ as

$$\big|\Phi(t,s)P^+(s)\big| \le K\,\mathrm{e}^{\alpha(t-s)}, \quad t\ge s, \qquad \big|\Phi(t,s)P^-(s)\big| \le K\,\mathrm{e}^{\beta(t-s)}, \quad t\le s$$

or

$$\big|P^+(t)\Phi(t,s)\big| \le K\,\mathrm{e}^{\alpha(t-s)}, \quad t\ge s, \qquad \big|P^-(t)\Phi(t,s)\big| \le K\,\mathrm{e}^{\beta(t-s)}, \quad t\le s$$

The notion of exponential dichotomy and its various basic applications is nicely presented for example in [14]. Note that the Definition 1.11 makes sense also for a complex Banach spaces.

Remark 1.3. Suppose that $X = \mathbb{K}^n$ with $\mathbb{K}^n = \mathbb{R}^n$ or $\mathbb{C}^n$. Let A be a constant squared matrix with n rows, we use notation $A \in M^{n\times n}(\mathbb{K})$. Now a sufficient condition for $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R})$ is

$$\{\lambda\in\mathbb{C} \,:\, \alpha \le \Re(\lambda) \le \beta\} \cap \sigma(A) = \emptyset$$

where $\Re$ is the real-part function and $\sigma(A)$ is the spectrum of A. Moreover $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R})$ still holds if we assume a weaker $\{\lambda\in\mathbb{C} \,:\, \alpha < \Re(\lambda) < \beta\} \cap \sigma(A) = \emptyset$ and that for any $\lambda\in\sigma(A)$ such that $\Re(\lambda)\in\{\alpha,\beta\}$ the corresponding complex Jordan block is a diagonal matrix λI. For details one should consult the books [35,41]).

We adopt the definition of *quasibounded* functions from [3, Definition 3.1]. We say that the interval I is unbounded to the left if I is one of the interval types $(-\infty, a)$, $(-\infty, a]$, $\mathbb{R}$, similarly I is unbounded to the right if I is one of the following interval types (a, ∞), $[a, \infty)$, $\mathbb{R}$.

Definition 1.12. Assume that I is unbounded to the left (resp. to the right). Let $g : I \to X$ be an arbitrary function and $\gamma \in \mathbb{R}$. We say that g is γ^--*quasibounded* (resp. γ^+-*quasibounded*) if $\|g\|^-_{\tau,\gamma} < \infty$ (resp. $\|g\|^+_{\tau,\gamma} < \infty$) for some $\tau \in I$, where

$$\|g\|^-_{\tau,\gamma} := \sup_{t\in\mathbb{R}^-_\tau} |g(t)|\, \mathrm{e}^{-\gamma t}, \qquad \left(\text{resp. } \|g\|^+_{\tau,\gamma} := \sup_{t\in\mathbb{R}^+_\tau} |g(t)|\, \mathrm{e}^{-\gamma t}\right)$$

In the peculiar $I = \mathbb{R}$ case we say that g is γ-*quasibounded* if $\|g\|_\gamma < \infty$ where

$$\|g\|_\gamma := \sup_{t\in\mathbb{R}} |g(t)|\, \mathrm{e}^{-\gamma t}.$$

We use the abbreviation *q. b.* for the long word "quasibounded".

Now we give a basic information about q. b. solutions of an inhomogeneous linear differential equations. Exponential dichotomy plays a crucial role.

Lemma 1.13. *Suppose that $\tau, \alpha, \beta, \gamma \in \mathbb{R}, \alpha < \gamma < \beta$ and $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R}^+_\tau)$ (resp. $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R}^-_\tau)$). Let us have a s. m. function $f : \mathbb{R}^+_\tau \to X$ (resp. $f : \mathbb{R}^-_\tau \to X$) such that $\|f\|^+_{\tau,\gamma} < \infty$ (resp. $\|f\|^-_{\tau,\gamma} < \infty$). Then $\mu : \mathbb{R}^+_\tau \to X$ (resp. $\mu : \mathbb{R}^-_\tau \to X$) is a γ^+-q. b. (resp. γ^--q. b.) solution of $\dot{x} = A(t)x + f(t)$ if and only if there is a $\xi^+ \in \mathbb{P}^+_\tau$ (resp. $\xi^- \in \mathbb{P}^-_\tau$) such that*

$$\left.\begin{aligned} \mu(t) = \Phi(t,\tau)\xi^+ + \int_\tau^t \Phi(t,s)P^+(s)f(s)\mathrm{d}s - \int_t^\infty \Phi(t,s)P^-(s)f(s)\mathrm{d}s \\ \left(\text{resp. } \mu(t) = \Phi(t,\tau)\xi^- + \int_\tau^t \Phi(t,s)P^-(s)f(s)\mathrm{d}s + \int_{-\infty}^t \Phi(t,s)P^+(s)f(s)\mathrm{d}s\right). \end{aligned}\right\} \tag{1.6}$$

In addition in both $\pm$ cases we have an estimation

$$\|\mu\|^\pm_{\tau,\gamma} \le K\, e^{-\gamma\tau}|\xi^\pm| + K\|f\|^\pm_{\tau,\gamma}\left(\frac{1}{\beta-\gamma} + \frac{1}{\gamma-\alpha}\right). \tag{1.7}$$

As a trivial consequence we get also that if $f : \mathbb{R} \to X$ is a s. m. function with $\|f\|_\gamma < \infty$ and $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R})$, then there is a unique γ-q. b. solution $y : \mathbb{R} \to X$ of $\dot{x} = A(t)x + f(t)$

and it is given by the formula

$$y(t) = \int_{-\infty}^{t} \Phi(t,s)P^+(s)f(s)\mathrm{d}s - \int_{t}^{\infty} \Phi(t,s)P^-(s)f(s)\mathrm{d}s. \tag{1.8}$$

Further we have

$$\|y\|_\gamma \le K\|f\|_\gamma \left(\frac{1}{\beta-\gamma} + \frac{1}{\gamma-\alpha}\right). \tag{1.9}$$

Proof. It is enough to study the "+ case" that is to characterize γ^+-q. b. solutions. Indeed, it is equally easy to perform the analogical proof of the "− case" (moreover the transformation $y(t) = x(-t)$ could be used to switch from "− to + case"). Having the representation (1.6) from (1.5) and $|f(s)| \le \|f\|^+_{\tau,\gamma}\, \mathrm{e}^{\gamma s}$ we infer easily (1.7). Similarly (1.9) follows from (1.8) which is again a consequence of (1.6). Hence it is enough to show (1.6) in the "+ case".

Let $\mu : \mathbb{R}^+_\tau \to X$ be a γ^+-q. b. solution of $\dot{x} = A(t)x + f(t)$. From (1.4) we have

$$\mu(t) = \Phi(t,t_0)\mu(t_0) + \int_{t_0}^{t} \Phi(t,s)f(s)\mathrm{d}s, \quad t, t_0 \ge \tau. \tag{1.10}$$

Using the projection $P^-(\tau)$ for both sides of equation with a rearrangement, setting $t = \tau$ we get easily

$$\begin{aligned}\left|P^-(\tau)\mu^+(\tau) + \int_{\tau}^{t_0} \Phi(\tau,s)P^-(s)f(s)\mathrm{d}s\right| &= \left|\Phi(\tau,t_0)P^-(t_0)\mu^+(t_0)\right| \\ &\le K\,\mathrm{e}^{\beta(\tau-t_0)}\|\mu\|^+_{\tau,\gamma}e^{\gamma t_0} = K\,\mathrm{e}^{\beta\tau}\|\mu\|^+_{\tau,\gamma}\,\mathrm{e}^{(\gamma-\beta)t_0}.\end{aligned}$$

Taking the limit as $t_0 \to \infty$ we get $P^-(\tau)\mu(\tau) = -\int_\tau^\infty \Phi(\tau,s)P^-(s)f(s)\mathrm{d}s$. Now set $\xi^+ := P^+(\tau)\mu(\tau)$. Using $\mu(\tau) = \xi^+ - \int_\tau^\infty \Phi(\tau,s)P^-(s)f(s)\mathrm{d}s$ and (1.10) with $t_0 = \tau$ we arrive at (1.6).

On the other hand let us have an arbitrary $\xi^+ \in \mathbb{P}^+_\tau$ and suppose that μ is given by (1.6). From Lemma 1.13 we have that $\Phi : \mathbb{R} \times \mathbb{R} \to \mathcal{B}(X)$ is continuous, therefore

$$\Phi(r,\cdot)P^-(\cdot) : \mathbb{R} \to \mathcal{B}(X)$$

is also continuous (for any $r \in \mathbb{R}$). Hence $g(s) := \Phi(r,s)P^-(s)f(s)$ is s. m. on $\mathbb{R}^+_r$. Moreover using (1.5) and that f is γ^+-quasibounded, it turns out that g is integrable on $\mathbb{R}^+_\tau$. We get similarly that

$$\int_t^\infty \Phi(t,s)P^-(s)f(s)\mathrm{d}s = \Phi(t,\tau)\int_\tau^\infty \Phi(\tau,s)P^-(s)f(s)\mathrm{d}s - \int_\tau^t \Phi(t,s)P^-(s)f(s)\mathrm{d}s.$$

So (1.10) is valid with $t_0 = \tau$. Using (1.4) we get that μ^+ is a solution and its γ^+-quasiboundedness is once again the consequence of (1.5) and $\|f\|^+_{\tau,\gamma} < \infty$. The proof is complete. $\square$

Applying Theorem 1.3 it is easy to extend this result to the Lipschitz nonlinearities. In the formulation we are using the standard double notation $\pm$.

Consequence 1.14. *Let us have again $\tau, \alpha, \gamma, \beta \in \mathbb{R}$ such that $\alpha < \gamma < \beta$ and $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R}^\pm_\tau)$. Suppose that $F \in \mathcal{CAR}(\mathbb{R}^\pm_\tau, X)$, $\|F(\cdot,0)\|^\pm_{\tau,\gamma} < \infty$ and*

$$|F(t,x_1) - F(t,x_2)| \le L|x_1 - x_2|, \quad t \in \mathbb{R}^\pm_\tau, x_1, x_2 \in X$$

for $L \ge 0$. Finally suppose that $KL\left(\frac{1}{\beta-\gamma} + \frac{1}{\gamma-\alpha}\right) < 1$. Then for every $\xi^\pm \in \mathbb{P}^\pm_\tau$ there is a unique $\gamma^\pm$-q. b. solution μ of $\dot{x} = A(t)x + F(t,x)$ such that $P^\pm(\tau)\mu(\tau) = \xi^\pm$.

Proof. We focus again only to the "+ case". Let μ be a γ^+-q. b. solution of $\dot{x} = A(t)x + F(t,x)$. Then μ is s. m. and $F(\cdot,\mu(\cdot))$ as well from Lemma 1.2. Moreover

$$\|F(\cdot,\mu(\cdot))\|^+_{\tau,\gamma} \le \|F(\cdot,0)\|^+_{\tau,\gamma} + L\|\mu\|^+_{\tau,\gamma} < \infty.$$

Now μ is a γ^+-q. b. solution of the linear equation $\dot{x} = A(t)x + F(t,\mu(t))$ hence Lemma 1.13 implies that

$$\mu(t) = \Phi(t,\tau)\xi^+ + \int_\tau^t \Phi(t,s)P^+(s)F(s,\mu(s))\mathrm{d}s - \int_t^\infty \Phi(t,s)P^-(s)F(s,\mu(s))\mathrm{d}s$$

for $\xi^+ \in \mathbb{P}^+_\tau$. Let us designate by $C^+_{\tau,\gamma}$ the Banach space of continuous and γ^+-q. b. functions from $\mathbb{R}^+_\tau$ to X endowed with a norm $\|\cdot\|^+_{\tau,\gamma}$. Consider an operator $\mathcal{T} : C^+_{\tau,\gamma} \times \mathbb{P}^+_\tau \to C^+_{\tau,\gamma}$ defined as

$$\mathcal{T}(x,\xi^+) := \Phi(t,\tau)\xi^+ + \int_\tau^t \Phi(t,s)P^+(s)F(s,x(s))\mathrm{d}s - \int_t^\infty \Phi(t,s)P^-(s)F(s,x(s))\mathrm{d}s.$$

Still from Lemma 1.13 follows the well-definiteness of $\mathcal{T}$. Note now that μ satisfies our theorem if and only if $T(\mu, \xi^+) = \mu$ for some ξ^+. The central property of $\mathcal{T}$ is its contractivity. For any $\eta^+ \in \mathbb{P}^+_\tau, x_1, x_2 \in C^+_{\tau,\gamma}$ using the $+$ variant of (1.7) with $\xi^+ := 0$ and $f(s) := F(s, x_1(s)) - F(s, x_2(s))$ we infer

$$\|\mathcal{T}(x_1, \eta^+) - \mathcal{T}(x_2, \eta^+)\|^+_{\tau,\gamma} \leq KL \left(\frac{1}{\beta - \gamma} + \frac{1}{\gamma - \alpha} \right) \|x_1 - x_2\|^+_{\tau,\gamma}.$$

Hence Theorem 1.3 is applicable (its remaining assumptions hold trivially) and its statement completes the proof. □

Remark 1.4. Let us suppose all the assumptions of the above consequence for two values $\gamma_{1,2}$ such that $\alpha < \gamma_1 < \gamma_2 < \beta$. Then for the corresponding $\gamma^+_{1,2}$-q. b. solutions μ_{γ_1,ξ^+} and μ_{γ_2,ξ^+} of

$$\dot{x} = A(t)x + F(t, x), \qquad P^+(\tau)x(\tau) = \xi^+$$

we have $\mu_{\gamma_1,\xi^+} = \mu_{\gamma_2,\xi^+}$. Indeed, for example in the "+ case" either of $\mu_{\gamma_1,\xi^+}, \mu_{\gamma_2,\xi^+}$ is a γ^+_2-q. b. solution with $P^+(\tau)\mu_{\gamma_1}(\tau) = P^+(\tau)\mu_{\gamma_2}(\tau) = \xi^+$. Hence the uniqueness part of Consequence 1.14 for $\gamma = \gamma_2$ implies our statement.

c

1.2 Inflated Dynamics with Exponential Dichotomy

From now on our inflated differential equation will be of the form

$$\dot{x} \in A(t)x + f(t,x) + g(t,x,X_1) \tag{1.11}$$

A central assumption in the whole section is $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R})$ for various $\alpha < \beta$. We apply classical tools to detect and characterize bounded/quasibounded solutions of (1.11) and corresponding *multivalued* stable/unstable integral manifolds.

1.2.1 Bounded Solutions

For this subsection suppose that

$$\alpha < 0 < \beta \text{ and } A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R}). \tag{1.12}$$

It is easy to see according to Remark 1.3 that (1.12) for suitable α, β is always satisfied for every constant matrix whose spectrum does not intersect the imaginary axis. Next we briefly recall the results from [12, Section 4, Theorem 3 and 4]. For the sake of completeness we also mention the main ideas of their proofs, again without technical details.

Theorem 1.15 (Theorem 3 in [12]). *Introduce* $\kappa_{\alpha,\beta} := \frac{1}{\beta} - \frac{1}{\alpha}$. *Let us have*

$$f : \mathbb{R} \times X \to X, \quad g : \mathbb{R} \times X \times X_1 \to X$$

such that

(i) Smoothness: $f(\cdot,x), g(\cdot,x,u) : \mathbb{R} \to X$ *are s. m. for all* $x \in X, u \in X_1$ *and* $f(t,\cdot) : X \to X, g(t,\cdot,\cdot) : X \times X_1 \to X$ *are continuous for all* $t \in \mathbb{R}$,

(ii) Boundedness: $|f(\cdot,0)|_\infty < \infty, |g(\cdot,0,0)|_\infty < \infty$,

(iii) Lipschitz condition: *there are constants* L_1, L_2, L_3 *such that*

$$|f(t,x_1) - f(t,x_2)| \le L_1|x_1 - x_2|,$$
$$|g(t,x_1,u_1) - g(t,x_2,u_2)| \le L_2|x_1 - x_2| + L_3|u_1 - u_2|$$

are valid for all $t \in \mathbb{R}, x_1, x_2 \in X, u_1, u_2 \in X_1$.

Finally suppose that

(iv) $K(L_1 + L_2)\kappa_{\alpha,\beta} < 1$.

Then for every $h \in \mathsf{H}_1$ *(see* (1.3)*) there exists a unique bounded solution* $\Gamma(\cdot, h) : \mathbb{R} \to X$ *of the problem* (1.11) *corresponding to the selector* h*. In addition the mapping* $\Gamma : \mathbb{R} \times \mathsf{H}_1 \to X$ *is continuous with an estimate*

$$|\Gamma(\cdot, h_1) - \Gamma(\cdot, h_2)|_\infty \le C(K, L_1, L_2, L_3, \alpha, \beta)|h_1 - h_2|_\infty$$

where

$$C(K, L_1, L_2, L_3, \alpha, \beta) := \frac{K L_3 \kappa_{\alpha,\beta}}{1 - K(L_1 + L_2)\kappa_{\alpha,\beta}}. \tag{1.13}$$

Proof. For $x \in C(\mathbb{R}, X), h \in \mathsf{H}_1$ and $t \in \mathbb{R}$ we set

$$\left.\begin{aligned} \mathcal{T}(x,h)(t) := \int_{-\infty}^{t} &\Phi(t,s)P^+(s)\big[f(s,x(s)) + g(s,x(s),h(s))\big]\mathrm{d}s \\ - \int_{t}^{\infty} &\Phi(t,s)P^-(s)\big[f(s,x(s)) + g(s,x(s),h(s))\big]\mathrm{d}s. \end{aligned}\right\} \tag{1.14}$$

Lemma 1.13 for $\gamma := 0$ implies that $\mathcal{T} : C(\mathbb{R}, X) \times \mathsf{H}_1 \to C(\mathbb{R}, X)$ is well-defined, continuous and that x is a bounded solution of $\dot{x} = A(t)x + f(t,x) + g(t,x,h(t))$ if and only if $\mathcal{T}(x,h) = x$. Moreover the uniform contractivity also follows from the mentioned lemma. Application of Theorem 1.3 gives our assertions, where $\Gamma(\cdot, h)$ is the unique fixed point of $\mathcal{T}(\cdot, h)$. □

For the later use we introduce another sets of selectors

$$\left.\begin{aligned} \mathsf{H}_\varepsilon &:= \{h \in \mathsf{H} : |h|_\infty \le \varepsilon\}, \\ \mathsf{H}^\pm_{\tau,\varepsilon} &:= \{h : \mathbb{R}^\pm_\tau \to X \text{ is s. m. and } |h|^\pm_\tau := \sup_{\pm(t-\tau)\ge 0} |h(t)| \le \varepsilon\}, \end{aligned}\right\} \tag{1.15}$$

where $\tau \in \mathbb{R}, \varepsilon \in [0,1]$. Note that $\mathsf{H}_\varepsilon, \mathsf{H}^{\pm}_{\tau,\varepsilon}$ are complete metric spaces with corresponding metrics derived naturally from norms $|\cdot|_\infty, |\cdot|^{\pm}_{\tau}$. Further in a situation of Theorem 1.15 we put down an important set of initial positions of the bounded solutions

$$S_{\tau,\varepsilon} := \{\Gamma(\tau,h) \,:\, h \in \mathsf{H}_\varepsilon\}, \qquad \varepsilon \in [0,1], \tau \in \mathbb{R}.$$

It is easy to see that every assumption of the fundamental Theorem 1.7 are satisfied in the context of the initial value problem

$$\dot{x} = A(t)x + f(t,x) + g(t,x,h(t)), \quad x(t_0) = x_0.$$

Hence we get the existence, uniqueness and continuous dependence of the solution $\lambda(\cdot,t_0,x_0,h)$ for every triple $(t_0,x_0,h) \in \mathbb{R} \times X \times \mathsf{H}_1$. Trivially, Theorem 1.7 is applicable also for intervals $I = \mathbb{R}^{\pm}_{\tau}$ and we get also solutions $\lambda^{\pm}_{\tau}(\cdot,t_0,x_0,h^{\pm})$ on $\mathbb{R}^{\pm}_{\tau}$ for $(t_0,x_0,h^{\pm}) \in \mathbb{R} \times X \times \mathsf{H}^{\pm}_{\tau,1}$. However the uniqueness of these solutions we have

$$\lambda(\cdot,t_0,x_0,h) = \lambda^{\pm}_{\tau}\left(\cdot,t_0,x_0,h|_{\mathbb{R}^{\pm}_{\tau}}\right) \text{ for any } h \in \mathsf{H}_1. \tag{1.16}$$

This argumentation justify that from now on we write in place of $\lambda^{\pm}_{\tau}$ simply λ.

For arbitrary functions $f : D_f \to X, g : D_g \to X$ we write $f \subset g$ if $D_f \subset D_g$ and $g|_{D_f} = f$. Now we define the stable set of $S_{\tau,\varepsilon}$ as

$$M^s_{\tau,\varepsilon} = M^s_{\tau,\varepsilon}(S_{\tau,\varepsilon}) := \Big\{\xi \in X \,:\, \exists h^+ \in \mathsf{H}^+_{\tau,\varepsilon}, h \in \mathsf{H}_\varepsilon \text{ such that } h^+ \subset h \text{ and } \lim_{t\to\infty} |\lambda(t,\tau,\xi,h^+) - \Gamma(t,h)| = 0\Big\},$$

and the unstable one as

$$M^s_{\tau,\varepsilon} = M^u_{\tau,\varepsilon}(S_{\tau,\varepsilon}) := \Big\{\xi \in X \,:\, \exists h^- \in \mathsf{H}^-_{\tau,\varepsilon}, h \in \mathsf{H}_\varepsilon \text{ such that } h^- \subset h \text{ and } \lim_{t\to-\infty} |\lambda(t,\tau,\xi,h^-) - \Gamma(t,h)| = 0\Big\}.$$

Further introduce

$$\widetilde{M}^s_{\tau,\varepsilon} := \left\{\xi \in X \,:\, \exists h^+ \in \mathsf{H}^+_{\tau,\varepsilon} \text{ such that } |\lambda(\cdot,\tau,\xi,h^+)|^+_\tau < \infty\right\},$$
$$\widetilde{M}^u_{\tau,\varepsilon} := \left\{\xi \in X \,:\, \exists h^- \in \mathsf{H}^-_{\tau,\varepsilon} \text{ such that } |\lambda(\cdot,\tau,\xi,h^-)|^-_\tau < \infty\right\}.$$

Note the straightforward relations $M^s_{\tau,\varepsilon} \subset \widetilde{M}^s_{\tau,\varepsilon}$, $M^u_{\tau,\varepsilon} \subset \widetilde{M}^u_{\tau,\varepsilon}$.

Theorem 1.16 (Theorem 4 in [12]). *Suppose all the assumptions of Theorem 1.15 and choose $\tau \in \mathbb{R}, \varepsilon \in [0,1]$. Then there are Lipschitz continuous functions*

$$w^s : \mathbb{P}^+_\tau \times \mathsf{H}^+_{\tau,\varepsilon} \to \mathbb{P}^-_\tau, \qquad w^u : \mathbb{P}^-_\tau \times \mathsf{H}^-_{\tau,\varepsilon} \to \mathbb{P}^+_\tau$$

such that

$$\left.\begin{aligned} M^s_{\tau,\varepsilon} = \widetilde{M}^s_{\tau,\varepsilon} = \left\{\xi^+ + w^s(\xi^+,h^+) \,:\, \xi^+ \in \mathbb{P}^+_\tau, h^+ \in \mathsf{H}^+_{\tau,\varepsilon}\right\},\\ M^u_{\tau,\varepsilon} = \widetilde{M}^u_{\tau,\varepsilon} = \left\{\xi^- + w^u(\xi^-,h^-) \,:\, \xi^- \in \mathbb{P}^-_\tau, h^- \in \mathsf{H}^-_{\tau,\varepsilon}\right\}. \end{aligned}\right\} \tag{1.17}$$

Exact Lipschitz constants are expressed in the formulaes

$$\left.\begin{aligned} &\left|w^s(\xi^+_1,h^+_1) - w^s(\xi^+_2,h^+_2)\right| \le \\ &\widetilde{C}(K,L_1,L_2,\alpha,\beta)|\xi^+_1 - \xi^+_2| + C(K,L_1,L_2,L_3,\alpha,\beta)|h^+_1 - h^+_2|^+_\tau,\\ &\left|w^u(\xi^-_1,h^-_1) - w^u(\xi^-_2,h^-_2)\right| \le \\ &\widetilde{C}(K,L_1,L_2,\alpha,\beta)|\xi^-_1 - \xi^-_2| + C(K,L_1,L_2,L_3,\alpha,\beta)|h^-_1 - h^-_2|^-_\tau, \end{aligned}\right\} \tag{1.18}$$

where C is defined in (1.13) *and $\widetilde{C} := \frac{K}{1-K(L_1+L_2)\kappa_{\alpha,\beta}}$.*

Proof. We give a short proof only for the stable case, the unstable variant is handled analogically. At first a characterisation of $\widetilde{M}^s_{\tau,\varepsilon}$ in (1.17) is proven and than the equality $\widetilde{M}^s_{\tau,\varepsilon} = M^s_{\tau,\varepsilon}$. Arguments from Lemma 1.13 yield that the suitable operator is

$$\mathcal{T}(x,\xi^+,h^+)(t) := \Phi(t,\tau)\xi^+ + \int_\tau^t \Phi(t,s)P^+(s)\Lambda_{x,h^+}(s)\mathrm{d}s - \int_t^\infty \Phi(t,s)P^-(s)\Lambda_{x,h^+}(s)\mathrm{d}s$$

viewed as $\mathcal{T} : C^+_\tau \times \mathbb{P}^+_\tau \times \mathsf{H}^+_{\tau,\varepsilon} \to C^+_\tau$, where $\Lambda_{x,h^+}(s) := f(s,x(s)) + g(s,x(s),h^+(s))$ and C^+_τ is the Banach space of bounded and continuous functions from $\mathbb{R}^+_\tau$ to X endowed with a norm $|\cdot|^+_\tau$. The operator $\mathcal{T}$ is suitable in a sense that $\mu \in C^+_\tau$ is a *bounded* solution of $\dot{x} = A(t)x + f(t,x) + g(t,x,h^+(t))$ if and only if $\mathcal{T}(\mu, \xi^+, h^+) = \mu$ for some $\xi^+ \in \mathbb{P}_\tau^+$.

Still from Lemma 1.13 we get that Theorem 1.3 is applicable. It shows the existence of the unique fixed point $\mathbf{x}_{\xi^+,h^+} \in C_\tau^+$ of $\mathcal{T}(\cdot, \xi^+, h^+)$ and also the corresponding Lipschitz bounds. The statement of (1.17) for $\widetilde{M}^s_{\tau,\varepsilon}$ then follows with $w^s(\xi^+, h^+) := \mathbf{x}_{\xi^+,h^+}(\tau)$.

For the nontrivial $\widetilde{M}^s_{\tau,\varepsilon} \subset M^s_{\tau,\varepsilon}$ we use Lemma 1.14. Let $\xi \in \widetilde{M}^s_{\tau,\varepsilon}$, then there is an $h^+ \in \mathsf{H}^+_{\tau,\varepsilon}$ such that $|\lambda(\cdot, \tau, \xi, h^+)|_\tau^+ < \infty$. Let us have $u : \mathbb{R}_\tau^+ \to X$ given by

$$u(t) = \lambda(t, \tau, \xi, h^+) - \Gamma(t, h), \qquad h \in \mathsf{H}_\varepsilon, h^+ \subset h.$$

Then u is a bounded solution (on $\mathbb{R}_\tau^+$) of $\dot{x} = A(t)x + F(t,x)$ on $\mathbb{R}_\tau^+$ where

$$F(t,x) := f(t, x + \Gamma(t,h)) - f(t, \Gamma(t,h)) + g(t, x + \Gamma(t,h), h^+(t)) - g(t, \Gamma(t,h), h^+(t)).$$

On the other hand $\|F(\cdot, 0)\|^+_{\tau,\gamma} = 0 < \infty$ for any $\gamma \in \mathbb{R}$ and $|F(t,x_1) - F(t,x_2)| \leq (L_1 + L_2)|x_1 - x_2|$. Finally choose $\gamma < 0$ such that $K(L_1 + L_2)\left(\frac{1}{\beta-\gamma} + \frac{1}{\gamma-\alpha}\right) < 1$, such a choice is possible due to the requirement (iv) of Theorem 1.15. Then every assumption of Lemma 1.14 is satisfied with $L := L_1 + L_2$. This yields a γ^+-q. b. solution v of $\dot{x} = A(t)x + F(t,x)$.

From Remark 1.4 we easily have $u = v$, hence $|u(t)| \leq \|v\|^+_{\tau,\gamma} \, \mathrm{e}^{\gamma t} \to 0$ as $t \to \infty$ and the proof is complete. Note that we get as a by-product the uniform exponential convergence in the definitions of the sets $M^s_{\tau,\varepsilon}, M^u_{\tau,\varepsilon}$ (uniformity for $h^+ \in \mathsf{H}^+_{\tau,\varepsilon}, h \in \mathsf{H}_\varepsilon, h^+ \subset h$ and $\tau \in \mathbb{R}$ – because γ does not depend on these quantities). □

Remark 1.5. An easy (in fact only formal) modification of the above proof yields Lipschitz functions

$$W^s : \mathbb{P}_\tau^+ \times \mathsf{H}_\varepsilon \to \mathbb{P}_\tau^-, \qquad W^u : \mathbb{P}_\tau^- \times \mathsf{H}_\varepsilon \to \mathbb{P}_\tau^+$$

such that

$$M^s_{\tau,\varepsilon} = \left\{\xi^+ + W^s(\xi^+, h) \,:\, \xi^+ \in \mathbb{P}_\tau^+, h \in \mathsf{H}_\varepsilon\right\},$$
$$M^u_{\tau,\varepsilon} = \left\{\xi^- + W^u(\xi^-, h) \,:\, \xi^- \in \mathbb{P}_\tau^-, h \in \mathsf{H}_e\right\}.$$

Their Lipschitz bounds (using $\widetilde{C}, C$ from (1.18)) are

$$\begin{aligned}\left|W^s(\xi_1^+, h_1) - W^s(\xi_2^+, h_2)\right| &\leq \widetilde{C}|\xi_1^+ - \xi_2^+| + C|h_1 - h_2|_\infty,\\ \left|W^u(\xi_1^-, h_1) - W^u(\xi_2^-, h_2)\right| &\leq \widetilde{C}|\xi_1^- - \xi_2^-| + C|h_1 - h_2|_\infty.\end{aligned}$$

Remark 1.6. Sets characterized in Theorem 1.16 have some invariance properties. Namely

$$\lambda(t, \tau, \xi^+ + W^s(\xi^+, h), h) \in M^s_{t+\tau,\varepsilon}, \qquad \lambda(t, \tau, \xi^- + W^u(\xi^-, h), h) \in M^u_{t+\tau,\varepsilon}$$

for every $t, \tau \in \mathbb{R}, \xi^\pm \times \mathbb{P}^\pm_\tau, h \in \mathsf{H}_\varepsilon$. Hence we sometimes refer to them as the *inflated* or *multivalued integral manifolds* of (1.2). Even though integral manifolds are usually defined as subsets of $\mathbb{R} \times X$ not only of X parameterized on $\mathbb{R}$. New sets

$$\begin{aligned}\mathcal{M}^s_\varepsilon &:= \{(\tau, \xi, h) \in \mathbb{R} \times X \times \mathsf{H}_\varepsilon \;:\; |\lambda(\cdot, \tau, \xi, h|^+_\tau < \infty\}\\ \mathcal{M}^u_\varepsilon &:= \{(\tau, \xi, h) \in \mathbb{R} \times X \times \mathsf{H}_\varepsilon \;:\; |\lambda(\cdot, \tau, \xi, h|^-_\tau < \infty\}\end{aligned}$$

would fix this problem we have for them

$$(\tau, \xi, h) \in \mathcal{M}^r_\varepsilon \quad \Rightarrow \quad (t + \tau, \lambda(t, \tau, \xi, h), h) \in \mathcal{M}^r_e, \qquad t \in \mathbb{R}, r \in \{s, u\}.$$

However from Theorem 1.16 we have (we describe only the stable case)

$$\begin{aligned}\mathcal{M}^s_\varepsilon &= \left\{\left(\tau, \xi^+ + W(\xi^+, h), h\right) \;:\; (\tau, \xi^+, h) \in \mathbb{R} \times \mathbb{P}^+_\tau \times \mathsf{H}_\varepsilon\right\}\\ &= \left\{\left(\tau, \widetilde{\Phi}(\tau)\xi_0^+ + W(\widetilde{\Phi}(\tau)\xi_0^+, h), h\right) \;:\; (\tau, \xi_0^+, h) \in \mathbb{R} \times \mathbb{P}^+_0 \times \mathsf{H}_\varepsilon\right\}\end{aligned}$$

where we have used that $\widetilde{\Phi}(\tau)|_{\mathbb{P}^+_0} : \mathbb{P}^+_0 \to \mathbb{P}^+_\tau$ is a bijective mapping (c.f. Definition 1.8). Hence for

$$\mathcal{W}^s : \mathbb{R} \times \mathbb{P}^+_0 \times \mathsf{H}_\varepsilon \to X, \quad \mathcal{W}^s(\tau, \xi_0^+, h) := \widetilde{\Phi}(\tau)\xi_0^+ + W(\widetilde{\Phi}(\tau)\xi_0^+, h)$$

we have a graph characterization

$$\mathcal{M}^s_\varepsilon = \left\{\left(\tau, \mathcal{W}(\tau, \xi_0^+, h), h\right) \;:\; (\tau, \xi_0^+, h) \in \mathbb{R} \times \mathbb{P}^+_0 \times \mathsf{H}_\varepsilon\right\}.$$

Note that $\mathcal{W}^s$ is uniformly Lipschitz in ξ_0^+ and h, however it is generally (due to our Carathédeory settings) only continuous in τ. Further we will not give remarks in this spirit (because it is more technical then interesting) and we restrict our attention to the τ-parametric sets $M^s_{\tau,\varepsilon}$, $M^u_{\tau,\varepsilon}$.

1.2.2 Quasibounded Solutions

Now we are going to investigate the set of quasibounded solutions of (1.11) and their stability/instability region. That is we are looking for quasibounded counterparts of Theorem 1.15 and 1.16. Our main tool will be the simple transformation discussed in the next lemma.

Lemma 1.17. *Assume that I is a positive interval and $f \in \mathcal{CAR}(I, X)$. Let $x : I \to X$ be a solution (in the sense of the Definition 1.4) of $\dot{x} = f(t, x)$ and $\rho : I \to \mathbb{R} \setminus \{0\}$ is a C^1 scalar function. Then $y : I \to X$ defined as $y(t) := \rho(t)x(t)$ is a solution of $\dot{y} = g(t, y)$ for $g(t, y) = \frac{\dot{\rho}(t)}{\rho(t)} y + \rho(t) f(t, \frac{1}{\rho(t)} y)$.*

Proof. Since x is a solution so $f(\cdot, x(\cdot)) : I \to X$ is locally integrable and $x(t) - x(s) = \int_s^t f(\tau, x(\tau)) \mathrm{d}\tau$ is valid for $t, s \in I$. Fix an arbitrary functional $\phi \in X^*$ and elements $t, s \in I$ such that $s < t$ (X^* is the dual space of X). With a notation $u(r) := \phi(x(r))$ we have $u(r_2) - u(r_1) = \int_{r_1}^{r_2} \phi(f(\tau, x(\tau))) \mathrm{d}\tau$ for $r_1, r_2 \in [s, t]$. Then $u : [s, t] \to \mathbb{R}$ is absolutely continuous on $[s, t]$. So $\dot{u}(r)$ exists for a. e. $r \in [s, t]$ and $\dot{u}(r) = \phi(f(r, x(r)))$ for these $r \in [s, t]$. This means that $v(r) := \rho(r)u(r)$ is also absolutely continuous on $[s, t]$ with derivative

$$\dot{v}(r) = \dot{u}(r)\rho(r) + u(r)\dot{\rho}(r) = \phi(f(r, x(r)))\rho(r) + \phi(x(r))\dot{\rho}(r)$$

for a. e. $r \in [s, t]$. Therefore for all $r_1, r_2 \in [s, t]$ we have

$$v(r_2) - v(r_1) = \int_{r_1}^{r_2} \Big(\phi(f(\tau, x(\tau)))\rho(\tau) + \phi(x(\tau))\dot{\rho}(\tau) \Big) \mathrm{d}\tau.$$

After elementary computations we obtain for all $\phi \in X^*$ and $s, t \in I$ the following equality

$$\phi\big(\rho(t)x(t) - \rho(s)x(s)\big) = \phi\left[\int_s^t \Big(\rho(\tau) f(\tau, x(\tau)) + \dot{\rho}(\tau)x(\tau) \Big) \mathrm{d}\tau \right]$$

or with a notation $y(t) = \rho(t)x(t)$

$$\phi\big(y(t) - y(s)\big) = \phi\left[\int_s^t \Big(\rho(\tau)f\big(\tau, \frac{1}{\rho(\tau)}y(\tau)\big) + \frac{\dot\rho(\tau)}{\rho(\tau)}y(\tau)\Big)\mathrm{d}\tau\right].$$

A consequence of the Hahn-Banach Theorem states that functionals separates points, so we have

$$y(t) - y(s) = \int_s^t \Big(\rho(\tau)f\big(\tau, \frac{1}{\rho(\tau)}y(\tau)\big) + \frac{\dot\rho(\tau)}{\rho(\tau)}y(\tau)\Big)\mathrm{d}\tau$$

which means exactly that $y : I \to X$ is a solution of $\dot y = g(t, y)$. □

Remark 1.7. This lemma shows in fact that it is possible to transform questions about γ-quasibounded solutions of $\dot x = A(t)x + F(t, x)$ to the task about the bounded solutions of $\dot y = (A(t) - \gamma I)y + \mathrm{e}^{-\gamma t}F(t, \mathrm{e}^{\gamma t}y)$. The right transformation between these problems is $y(t) = x(t)\,\mathrm{e}^{-\gamma t}$ and for justification it is enough to apply Lemma 1.17 with $\rho(t) := \mathrm{e}^{-\gamma t}$.

Noting the bijective correspondence between these solution sets and the fact

$$(A(t) - \gamma I) \in \mathcal{E}_{\gamma-\alpha,\beta-\gamma}(I)$$

with $\gamma - \alpha < 0 < \beta - \gamma$, we can generalize Theorem 1.15 as follows.

Theorem 1.18. *Assume that we have functions $f : \mathbb{R} \times X \to X, g : \mathbb{R} \times X \times X_1 \to X$ and a constant $\gamma \in (\alpha, \beta)$ such that*

(i) Smoothness: $f(\cdot, x), g(\cdot, x, u) : \mathbb{R} \to X$ *are s. m. for all* $x \in X, u \in X_1$ *and* $f(t, \cdot) : X \to X, g(t, \cdot, \cdot) : X \times X_1 \to X$ *are continuous for all* $t \in \mathbb{R}$,

(ii) Quasiboundedness: $\|f(t, 0)\|_\gamma < \infty, \|g(t, 0, 0)\|_\gamma < \infty$,

(iii) Lipschitz condition: *there are constants* L_1, L_2, L_3 *such that*

$$|f(t, x_1) - f(t, x_2)| \le L_1|x_1 - x_2|$$

and

$$|g(t, x_1, u_1) - g(t, x_2, u_2)| \le L_2|x_1 - x_2| + L_3\, e^{\gamma t}|u_1 - u_2| \tag{1.19}$$

are valid for all $t \in \mathbb{R}, x_1, x_2 \in X, u_1, u_2 \in X_1$,

(iv) $K(L_1 + L_2)\kappa_{\alpha-\gamma,\beta-\gamma} < 1$.

Then for every $h \in \mathsf{H}_1$ *there exists a unique* γ*-q. b. solution* $\Gamma_\gamma(\cdot, h) : \mathbb{R} \to X$ *of the problem* (1.11) *corresponding to the selector* h. *In addition the mapping* $\Gamma_\gamma : \mathbb{R} \times \mathsf{H}_1 \to X$ *is continuous and*

$$\|\Gamma_\gamma(\cdot, h_1) - \Gamma_\gamma(\cdot, h_2)\|_\gamma \le C(K, L_1, L_2, L_3, \alpha - \gamma, \beta - \gamma)|h_1 - h_2|_\infty,$$

where C *is given in* (1.13).

Proof. Apply the transformation $y(t) = \varrho(t)x(t)$ from Lemma 1.17 with $\varrho(t) := \mathrm{e}^{-\gamma t}$. Then the differential inclusion for y is

$$\dot{y} \in \tilde{A}(t)y + \tilde{f}(t, y) + \tilde{g}(t, y, X_1) \tag{1.20}$$

where

$$\tilde{A}(t) = A(t) - \gamma I \in \mathcal{E}_{\alpha-\gamma,\beta-\gamma}(\mathbb{R}), \qquad \alpha - \gamma < 0 < \beta - \gamma,$$
$$\tilde{f}(t, y) = \mathrm{e}^{-\gamma t} f(t, \mathrm{e}^{\gamma t} y), \qquad \tilde{g}(t, y, u) = \mathrm{e}^{-\gamma t} g(t, \mathrm{e}^{\gamma t} y, u).$$

Assumptions (i)–(iv) of Theorem 1.15 are direct counterparts of (i)–(iv) listed above. Hence, Theorem 1.15 ensures the well-definiteness of the solution operator $\tilde{\Gamma} : \mathbb{R} \times \mathsf{H}_1 \to X$ of bounded solutions of (1.20). With a notation $\Gamma_\gamma(t, h) := \mathrm{e}^{\gamma t}\tilde{\Gamma}(t, h)$ we arrive at the statement and the proof is finished. □

Remark 1.8. At first glance the Lipschitz-exponential condition (1.19) is quite unnatural. To avoid it let us introduce a new set of selectors

$$\mathsf{H}_\varepsilon^\gamma := \{h \in \mathsf{H}_\varepsilon : \|h\|_\gamma < \infty\}, \qquad \varepsilon \in [0, 1].$$

$\mathsf{H}_\varepsilon^\gamma$ endowed with a metric $d_\gamma(h_1, h_2) := \|h_1 - h_2\|_\gamma$ is a complete metric space (the topology on $\mathsf{H}_\varepsilon^\gamma$ is induced by this metric). Now suppose all the conditions of Theorem 1.18 and replace (1.19) by the usual

$$|g(t, x_1, u_1) - g(t, x_2, u_2)| \le L_2|x_1 - x_2| + L_3|u_1 - u_2|. \tag{1.21}$$

In this framework we have to modify our the assertion of our theorem as follows. Corresponding to any $h \in \mathsf{H}_1^\gamma$ there is a unique γ-q. b. solution $\widehat{\Gamma}_\gamma(\cdot, h) : \mathbb{R} \to X$ of (1.11). $\widehat{\Gamma}_\gamma : \mathbb{R} \times \mathsf{H}_1^\gamma \to X$ is continuous and

$$\|\widehat{\Gamma}_\gamma(\cdot, h_1) - \widehat{\Gamma}_\gamma(\cdot, h_2)\|_\gamma \leq C(K, L_1, L_2, L_3, \alpha - \gamma, \beta - \gamma)\|h_1 - h_2\|_\gamma.$$

To justify this statement let us introduce $\mathcal{K}_\gamma : \mathsf{H}_1^\gamma \to \mathsf{H}_1$ as $\mathcal{K}_\gamma(h)(t) := \mathrm{e}^{-\gamma t}h(t)$ and $\widetilde{g}(t, x, u) := g(t, x, \mathrm{e}^{\gamma t}u)$. Then

$$\dot{x} \in A(t)x + f(t, x) + \widetilde{g}(t, x, X_1) \tag{1.22}$$

fulfills trivially all the prerequisites of Theorem 1.18 (even the critical (1.19)). The application of this theorem gives $\Gamma_\gamma^*(\cdot, \cdot) : \mathbb{R} \times \mathsf{H}_1 \to X$ the γ-q. b. solution operator of (1.22). Returning to our original setting let us introduce $\widehat{\Gamma}_\gamma : \mathbb{R} \times \mathsf{H}_1^\gamma \to X$ as $\widehat{\Gamma}_\gamma(t, h) := \Gamma_\gamma^*(t, \mathcal{K}(h))$. Then the statement follows easily, for example the Lipschitz property comes from

$$\begin{aligned}
\left\|\widehat{\Gamma}_\gamma(\cdot, h_1) - \widehat{\Gamma}_\gamma(\cdot, h_2)\right\| &= \left\|\Gamma_\gamma^*(\cdot, \mathcal{K}(h_1)) - \Gamma_\gamma^*(\cdot, \mathcal{K}(h_2))\right\| \\
&\leq C(K, L_1, L_2, L_3, \alpha - \gamma, \beta - \gamma)|\mathcal{K}(h_1) - \mathcal{K}(h_2)|_\infty \\
&= C(K, L_1, L_2, L_3, \alpha - \gamma, \beta - \gamma)\|h_1 - h_2\|_\gamma.
\end{aligned}$$

The conclusion of this remark is that if one insists on the natural Lipschitz condition then its price has to be paid in a form of reducing our selector space.

It is clear that Theorem 1.7 under the assumptions of Theorem 1.18 implies that the problem (1.11) is uniquely solvable on $\mathbb{R}$ corresponding to selector $h \in \mathsf{H}_1$ with initial condition $x(\tau) = \xi$. If we designate this solution by $\lambda = \lambda(t, \tau, \xi, h)$ then $\lambda : \mathbb{R} \times \mathbb{R} \times X \times \mathsf{H}_1 \to X$ is continuous. We will use the same notation λ also for selectors defined only on $\mathbb{R}_\tau^\pm$, we are aware of the arising ambiguity, see the discussion around (1.16).

Now we generalize Theorem 1.16. Suppose that all the conditions of Theorem

(1.18) is fulfilled and let us introduce

$$
\begin{aligned}
S^{\gamma}_{\tau,\varepsilon} &:= \{\Gamma_\gamma(\tau,h)\,:\,h\in \mathsf{H}_\varepsilon\},\\
M^{s,\gamma}_{\tau,\varepsilon} &:= \{\xi\in X\,:\,\exists h^+\in \mathsf{H}^+_{\tau,\varepsilon}, \exists h\in \mathsf{H}_\varepsilon, h^+\subset h \text{ and}\\
&\qquad \lim_{t\to\infty}\|\lambda(t,\tau,\xi,h^+)-\Gamma_\gamma(t,h)\|\,\mathrm{e}^{-\gamma t}=0\},\\
M^{u,\gamma}_{\tau,\varepsilon} &:= \{\xi\in X\,:\,\exists h^-\in \mathsf{H}^-_{\tau,\varepsilon}, \exists h\in \mathsf{H}_\varepsilon, h^-\subset h \text{ and}\\
&\qquad \lim_{t\to-\infty}\|\lambda(t,\tau,\xi,h^-)-\Gamma_\gamma(t,h)\|\,\mathrm{e}^{-\gamma t}=0\}.
\end{aligned}
$$

These notations might be somewhat misleading, note that the letters s, u no longer refer to the stable and unstable case in a usual sense that is $\lim_{t\to\pm\infty}\|\lambda(t,\tau,\xi,h^\pm)-\Gamma_\gamma(t,h)\| = 0$. It still has a reason to use $M^{s,\gamma}_{\tau,\varepsilon}, M^{u,\gamma}_{\tau,\varepsilon}$, firstly because it reveals the connection between the theory of q. b. and bounded solutions. Secondly because in an appropriate exponential rate s and u still refer to some asymptotic behavior.

Theorem 1.19. *Suppose all the assumptions of Theorem 1.18 and fix $\tau\in\mathbb{R}, \varepsilon\in[0,1]$. Then there are Lipschitz continuous functions*

$$
w^{s,\gamma}:\mathbb{P}^+_\tau\times \mathsf{H}^+_{\tau,\varepsilon}\to\mathbb{P}^-_\tau,\quad w^{u,\gamma}:\mathbb{P}^-_\tau\times \mathsf{H}^-_{\tau,\varepsilon}\to\mathbb{P}^+_\tau
$$

such that

$$
\begin{aligned}
M^{s,\gamma}_{\tau,\varepsilon} &= \{\xi\in X\,:\,\exists h^+\in \mathsf{H}^+_{\tau,\varepsilon}\,:\,\|\lambda(\cdot,\tau,\xi,h^+)\|^+_{\tau,\gamma}<\infty\}\\
&= \{\xi^+ + w^{s,\gamma}(\xi^+,h)\,:\,\xi^+\in\mathbb{P}^+_\tau, h\in \mathsf{H}^+_{\tau,\varepsilon}\},\\
M^{u,\gamma}_{\tau,\varepsilon} &= \{\xi\in X\,:\,\exists h^-\in \mathsf{H}^-_{\tau,\varepsilon}\,:\,\|\lambda(\cdot,\tau,\xi,h^-)\|^-_{\tau,\gamma}<\infty\}\\
&= \{\xi^- + w^{u,\gamma}(\xi^-,h),:\,\xi^-\in\mathbb{P}^-_\tau, h\in \mathsf{H}^-_{\tau,\varepsilon}\}.
\end{aligned}
$$

Moreover we get exact Lipschitz bounds if we replace in (1.18) w^s, w^u, α, β *by* $w^{s,\gamma}$, $w^{u,\gamma}$, $\alpha-\gamma, \beta-\gamma$.

Proof. We present here the proof of the "stable" case, the other one is similar. We follow the lines of the proof of Theorem 1.18 and we are going to use notations therein. Denote by $\tilde{\lambda}(\cdot,\tau,\xi,h)$ the solution of the transformed inclusion (1.20) on $\mathbb{R}$ (resp. $\mathbb{R}^\pm$) with initial condition $y(\tau)=\xi$. Theorem 1.16 for an appropriate function $\widetilde{w}^s:\mathbb{P}^+_\tau\times \mathsf{H}^+_{\tau,\varepsilon}\to\mathbb{P}^-_\tau$ (continuous and satisfying (1.18) where α,β are replaced by

$\alpha - \gamma$ and $\beta - \gamma$) yields

$$\{\xi \in X, \exists h^+ \in \mathsf{H}^+_{\tau,\varepsilon}, \exists h \in \mathsf{H}_\varepsilon, h^+ \subset h, \lim_{t\to\infty} |\tilde{\lambda}(t,\tau,\xi,h^+) - \tilde{\Gamma}(t,h)| = 0\} =$$

$$\{\xi \in X, \exists h^+ \in \mathsf{H}^+_{\tau,\varepsilon}, |\tilde{\lambda}(t,\tau,\xi,h^+)|^+_\tau < \infty\} =$$

$$\{\xi^+ + \widetilde{w}^s(\xi^+,h^+), \xi^+ \in \mathbb{P}^+_\tau, h^+ \in \mathsf{H}^+_{\tau,\varepsilon}\}.$$

This identity implies exactly the statement of the theorem. It is enough to set $w^{s,\gamma} := \widetilde{w}^s$ and to the relations $\tilde{\lambda} = \lambda\, \mathrm{e}^{-\gamma t}$ and $\tilde{\Gamma} = \Gamma_\gamma\, \mathrm{e}^{-\gamma t}$. The proof is finished. □

Remark 1.9. The theorem above has again a variant for the situation when (1.19) is replaced by (1.21). Now it is straightforward how to achieve this, we will use notations from Remark 1.8. Let us begin with new selector spaces

$$\mathsf{H}^{\pm;\gamma}_{\tau,\varepsilon} := \{h \in \mathsf{H}^\pm_{\tau,\varepsilon} \;:\; \|h^\pm\|^\pm_{\tau,\gamma} < \infty\}.$$

Endowing $\mathsf{H}^{\pm;\gamma}_{\tau,\varepsilon}$ with metrics $d^\pm_{\tau,\gamma}(h_1,h_2) := \|h_1 - h_2\|^\pm_{\tau,\gamma}$ they turn into complete metric spaces.

This time we have to apply the fundamental Theorem 1.5 to get a continuous $\widehat{\lambda} : \mathbb{R}^\pm \times \mathbb{R} \times X \times \mathsf{H}^\gamma_\varepsilon \to X$ (or $\widehat{\lambda} : \mathbb{R}^\pm_\tau \times \mathbb{R}^\pm_\tau \times X \times \mathsf{H}^{\pm;\gamma}_{\tau,\varepsilon} \to X$ with a same notation $\widehat{\lambda}$) such that $\widehat{\lambda}(\cdot,\tau,\xi,h)$ is a unique solution of $\dot{x} = A(t)x + f(t,x) + g(t,x,h(t))$ with $x(\tau) = \xi$ on $\mathbb{R}$ (resp. $\mathbb{R}^\pm_\tau$). Let us have $\mathcal{K}^\pm_\gamma : \mathsf{H}^{\pm;\gamma}_{\tau,\varepsilon} \to \mathsf{H}^\pm_{\tau,\varepsilon}$ the transformation given by $\mathcal{K}^\pm_\gamma(h)(t) := \mathrm{e}^{-\gamma t} h(t)$.

Introduce

$$\begin{aligned}
\widehat{S}^\gamma_{\tau,\varepsilon} &:= \big\{\widehat{\Gamma}_\gamma(\tau,h) \;:\; h \in \mathsf{H}^\gamma_\varepsilon\big\},\\
\widehat{M}^{s,\gamma}_{\tau,\varepsilon} &:= \big\{\xi \in X \;:\; \exists h^+ \in \mathsf{H}^{+;\gamma}_{\tau,\varepsilon}, \exists h \in \mathsf{H}^\gamma_\varepsilon, h^+ \subset h \text{ such that}\\
&\qquad \lim_{t\to\infty} \|\lambda(t,\tau,\xi,h^+) - \widehat{\Gamma}_\gamma(t,h)\|\, \mathrm{e}^{-\gamma t} = 0\big\},\\
\widehat{M}^{u,\gamma}_{\tau,\varepsilon} &:= \big\{\xi \in X \;:\; \exists h^- \in \mathsf{H}^{-;\gamma}_{\tau,\varepsilon}, \exists h \in \mathsf{H}^\gamma_\varepsilon, h^- \subset h \text{ such that}\\
&\qquad \lim_{t\to-\infty} \|\lambda(t,\tau,\xi,h^-) - \widehat{\Gamma}_\gamma(t,h)\|\, \mathrm{e}^{-\gamma t} = 0\big\},
\end{aligned}$$

For modified system (1.22) the Theorem 1.19 is applicable and gives functions $(w^*)^{s,\gamma}$, $(w^*)^{u,\gamma}$ with corresponding properties. Then for Lipschitz continuous functions

$$\widehat{w}^{s,\gamma}:\mathbb{P}^+_\tau\times \mathsf{H}^{\gamma,+}_{\tau,\varepsilon}\to\mathbb{P}^-_\tau,\quad \widehat{w}^{u,\gamma}:\mathbb{P}^-_\tau\times \mathsf{H}^{\gamma,-}_{\tau,\varepsilon}\to\mathbb{P}^+_\tau$$

defined as

$$\widehat{w}^{s,\gamma}(\xi^+,h^+):=(w^*)^{s,\gamma}(\xi^+,\mathcal{K}^+_\gamma(h^+)),\quad \widehat{w}^{u,\gamma}(\xi^-,h^-):=(w^*)^{u,\gamma}(\xi^-,\mathcal{K}^-_\gamma(h^-))$$

we have

$$\begin{aligned}\widehat{M}^{s,\gamma}_{\tau,\varepsilon}&=\{\xi\in X\,:\,\exists h^+\in\mathsf{H}^{+,\gamma}_{\tau,\varepsilon}\,:\,\|\widehat{\lambda}(\cdot,\tau,\xi,h^+)\|^+_{\tau,\gamma}<\infty\}\\&=\{\xi^++\widehat{w}^{s,\gamma}(\xi^+,h^+)\,:\,\xi^+\in\mathbb{P}^+_\tau,h^+\in\mathsf{H}^{+,\gamma}_{\tau,\varepsilon}\},\\\widehat{M}^{u,\gamma}_{\tau,\varepsilon}&=\{\xi\in X\,:\,\exists h^-\in\mathsf{H}^{-,\gamma}_{\tau,\varepsilon}\,:\,\|\widehat{\lambda}(\cdot,\tau,\xi,h^-)\|^-_{\tau,\gamma}<\infty\}\\&=\{\xi^-+\widehat{w}^{u,\gamma}(\xi^-,h^-),:\,\xi^-\in\mathbb{P}^-_\tau,h^-\in\mathsf{H}^{-,\gamma}_{\tau,\varepsilon}\}.\end{aligned}$$

The Lipschitz bounds – for instance in the "stable" case – follows from

$$\begin{aligned}\left|\widehat{w}^{s,\gamma}(\xi^+_1,h^+_1)-\widehat{w}^{s,\gamma}(\xi^+_2,h^+_2)\right|&=\left|(w^*)^{s,\gamma}(\xi^+_1,\mathcal{K}^+_\gamma(h^+_1))-(w^*)^{s,\gamma}(\xi^+_2,\mathcal{K}^+_\gamma(h^+_2))\right|\\\le\widetilde{C}|\xi^+_1-\xi^+_2|+C|\mathcal{K}^+_\gamma(h^+_1)-\mathcal{K}^+_\gamma(h^+_2)|^+_\tau&=\widetilde{C}|\xi^+_1-\xi^+_2|+C\|h^+_1-h^+_2\|^+_{\tau,\gamma}\end{aligned}$$

where $\widetilde{C}$ and C come from (1.18) as it is described in the formulation of Theorem 1.19.

1.2.3 Independence and Hierarchy

Now we change the topic and our following short investigations will be focused on some interesting questions concerning the previously defined and characterized notions. Without any pressure and efforts to be as complete as possible we rather would like to make a starting point for possible further directions of research. We show some results basically motivated by two questions:

Q1: Under which conditions are we able to prove the independence of Γ_γ on γ?

Q2: What relations should we expect between various integral manifolds described in Subsection 1.2.2 if the linear part possesses exponential dichotomy on $\mathbb{R}$ corresponding to more then one, properly linked projection?

At first partial answer to the question **Q1.**

Theorem 1.20. *Let us have $\alpha < \alpha_1 < \beta_1 < \beta$ and functions*

$$f : \mathbb{R} \times X \to X, \quad g : \mathbb{R} \times X \times X_1 \to X$$

such that

(i) Smoothness: *$f(\cdot, x), g(\cdot, x, u) : \mathbb{R} \to X$ are s. m. for all $x \in X, u \in X_1$ and $f(t, \cdot) : X \to X, g(t, \cdot, \cdot) : X \times X_1 \to X$ are continuous for all $t \in \mathbb{R}$,*

(ii) Upper bound: *there are constants $M_1, M_2 \geq 0$ such that*

$$|f(t, 0)| \leq M_1 \eta(t), \quad |g(t, 0, 0)| \leq M_2 \eta(t), \qquad t \in \mathbb{R},$$

where $\eta(t) := \min\{ e^{\alpha_1 t}, e^{\beta_1 t}\}$,

(iii) Lipschitz condition: *there are constants $L_1, L_2, L_3 \geq 0$ such that*

$$|f(t, x_1) - f(t, x_2)| \leq L_1 |x_1 - x_2|,$$
$$|g(t, x_1, u_1) - g(t, x_1, u_2)| \leq L_2 |x_1 - x_2| + L_3 \eta(t) |u_1 - u_2|$$

are valid for all $t \in \mathbb{R}, x_1, x_2 \in X, u_1, u_2 \in X_1$,

(iv) for a constant $\theta := \max \left\{ \kappa_{\alpha-\alpha_1, \beta-\alpha_1}, \kappa_{\alpha-\beta_1, \beta-\beta_1} \right\}$ we have $K(L_1 + L_2)\theta < 1$.

Then Γ_γ from Theorem 1.18 is well-defined for $\gamma \in [\alpha_1, \beta_1]$ and independent from γ – that is $\Gamma_{\gamma_1} = \Gamma_{\gamma_2}$ for all $\gamma_1, \gamma_2 \in [\alpha_1, \beta_1]$.

Proof. We set $|x|_{[\alpha_1, \beta_1]} := \|x\|_{\alpha_1} + \|x\|_{\beta_1}$ for $x \in C(\mathbb{R}, X)$. The space

$$Y := \{x \in C(\mathbb{R}, X) \, : \, |x|_{[\alpha_1, \beta_1]} < \infty\}$$

become a Banach space with a norm $| \cdot |_{[\alpha_1, \beta_1]}$. Define $\mathcal{T} : Y \times \mathsf{H}_1 \to Y$ formally as in (1.14). Fix $x \in Y, h \in \mathsf{H}_1$ and introduce $\tilde{f}(s) := f(s, x(s)) + g(s, x(s), h(s))$. From our assumptions we infer for $\gamma \in \{\alpha_1, \beta_1\}$ that

$$\|\tilde{f}\|_\gamma \leq (L_1 + L_2)\|x\|_\gamma + L_3 |h|_\infty + \|f(\cdot, 0)\|_\gamma + \|g(\cdot, 0, 0)\|_\gamma < \infty.$$

Hence Lemma 1.13 implies that $\mathcal{T}(x,h)$ is the unique γ-q. b. solution of $\dot{y} = A(t)y + \tilde{f}$. Using (1.9) (again for $\gamma \in \{\alpha, \beta\}$) we get the well-definiteness of $\mathcal{T}$. Moreover $\mathcal{T}$ is continuous and also a uniform (in H_1) contraction (on Y), this follows from the easily derivable estimations

$$|\mathcal{T}(x,h_1) - \mathcal{T}(x,h_2)|_{[\alpha_1,\beta_1]} \leq KL_3(\kappa_{\alpha-\alpha_1,\beta-\alpha_1} + \kappa_{\alpha-\beta_1,\beta-\beta_1})|h_1 - h_2|_\infty,$$
$$|\mathcal{T}(x_1,h) - \mathcal{T}(x_2,h)|_{[\alpha_1,\beta_1]} \leq K(L_1 + L_2)\theta|x_1 - x_2|_{[\alpha_1,\beta_1]}.$$

Theorem 1.3 is applicable, it yields the unique fix point x of $\mathcal{T}(\cdot,h)$ in the space Y (note that x is a solution of the nonlinear problem $\dot{y} = A(t)y + f(t,y) + g(t,y,h(t))$). Using the uniqueness property of Γ_γ from Theorem 1.18 and that for every $y \in Y$ and $\gamma \in [\alpha_1, \beta_1]$ we have $\|y\|_\gamma < \infty$ we infer that $\Gamma_\gamma = \mathrm{x}$ for all $\gamma \in [\alpha_1, \beta_1]$ and we are done. □

Note that in the spirit of the theorem above without any effort we should develop γ-independent variants of Theorems 1.18, 1.19 and Remarks 1.8, 1.9. Because it would be mechanical we do not present it and turn our attention to the question **Q2**.

Let us have for $i = 1, \cdots, n, n \geq 2$, projections $P_i \in \mathcal{B}(X)$ (that is $P_i^2 = P_i$) and $K_i \geq 1, \alpha_i, \beta_i, \gamma_i \in \mathbb{R}$. Suppose that

$$\alpha_i < \gamma_i < \beta_i, \quad i = 1, \cdots, n \text{ and } \beta_i \leq \alpha_{i+1}, \quad i = 1, \cdots, n-1.$$

Set $P_i^+ := P_i, P_i^- := \mathbb{I} - P_i$ and assume the following *hierarchy* of the projector ranges

$$P_i^+(X) \subset P_{i+1}^+(X), \quad P_i^-(X) \supset P_{i+1}^-(X), \qquad i = 1, \cdots, n-1. \tag{1.23}$$

Now suppose $A \in \mathcal{E}_{\alpha_i,\beta_i}(X) = \mathcal{E}_{\alpha_i,\beta_i}(X; P_i, K_i)$ that is

$$\left|\Phi(t,0)P_i^+\Phi(0,s)\right| \leq K_i\, \mathrm{e}^{\alpha_i(t-s)}, \quad t \geq s,$$
$$\left|\Phi(t,0)P_i^-\Phi(0,s)\right| \leq K_i\, \mathrm{e}^{\beta_i(t-s)}, \quad t \leq s.$$

Introduce moreover $\eta^*(t) := \min_{i=1,\cdots,n}\{\,\mathrm{e}^{\gamma_i t}\}$ and $\theta^* := \max_{i=1,\cdots,n}\{\kappa_{\alpha_i-\gamma_i,\beta_i-\gamma_i}\}$. Under these assumptions we can state the following theorem.

Theorem 1.21. *Let us have* $f : \mathbb{R} \times X \to X$, $g : \mathbb{R} \times X \times X_1 \to X$ *such that*

(i) Smoothness: $f(\cdot, x), g(\cdot, x, u) : \mathbb{R} \to X$ *are s. m. for all* $x \in X, u \in X_1$ *and* $f(t, \cdot) : X \to X, g(t, \cdot, \cdot) : X \times X_1 \to X$ *are continuous for all* $t \in \mathbb{R}$,

(ii) Upper bound: *there is a constants* $M \geq 0$ *such that*

$$|f(t,0)| \leq M\eta^*(t), \quad |g(t,0,0)| \leq M\eta^*(t), \qquad t \in \mathbb{R},$$

(iii) Lipschitz condition: *there are constants* L_1, L_2, L_3 *such that*

$$|f(t,x_1) - f(t,x_2)| \leq L_1|x_1 - x_2|$$
$$|g(t,x_1,u_1) - g(t,x_1,u_2)| \leq L_2|x_1 - x_2| + L_3\eta^*(t)|u_1 - u_2|$$

are valid for all $t \in \mathbb{R}, x_1, x_2 \in X, u_1, u_2 \in X_1$,

(iv) *we have* $K(L_1 + L_2)\theta^* < 1$.

Then $\Gamma_{\gamma_i}, M^{s,\gamma_i}_{\tau,\varepsilon}, M^{u,\gamma_i}_{\tau,\varepsilon}$ *from Theorems 1.18 , 1.19 concerning the inflated differential equation* $\dot{x} \in A(t)x + f(t,x) + g(t,x,X_1)$ *are well-defined and the following inherited (from* (1.23)*) hierarchy is valid*

$$M^{s,\gamma_i}_{\tau,\varepsilon} \subset M^{s,\gamma_{i+1}}_{\tau,\varepsilon}, \qquad M^{u,\gamma_i}_{\tau,\varepsilon} \supset M^{u,\gamma_{i+1}}_{\tau,\varepsilon}, \qquad i = 1, \cdots, n-1.$$

Proof. We present the proof of the "stable" variant. In fact the statement follows immediately from (cf. Theorem 1.16)

$$M^{s,\gamma_i}_{\tau,\varepsilon} = \left\{ \xi \in X \ : \ \|\lambda(\cdot, \tau, \xi, h)\|^+_{\tau,\gamma_i} < \infty \right\}, \quad i = 1, \cdots, n$$

combined with an elementary fact

$$\|\cdot\|^+_{\tau,\gamma_i} \geq \|\cdot\|^+_{\tau,\gamma_{i+1}} \, \mathrm{e}^{(\gamma_{i+1}-\gamma_i)\tau}, \quad i = 1, \cdots, n-1.$$

The proof is complete. □

The hierarchy of integral manifolds for non-autonomous systems without inflation and with a bit restrictive $f(t,0) = 0$ was brilliantly presented in [3,4].

1.3 Remarks on Exponential Dichotomy of ODE

Here we would like to present simple criteria on *hyperbolic* exponential dichotomy for linear systems. That is there are $\alpha < 0 < \beta$ such that $A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R})$. In this section by exponential dichotomy we always mean this property. This section is essentially independent from previous ones, hence there might be some duplicity in notations which will be always clarified. We will use the following well-known lemma (cf. [61]).

Lemma 1.22 (Neumann's Inversion Lemma)**.** *Suppose that X is a Banach space and $A \in \mathcal{B}(X)$ is invertible. Then for $B \in \mathcal{B}(X)$ such that $|A^{-1}B| < 1$ we have $(A+B)^{-1} \in \mathcal{B}(X)$, and*

$$(A+B)^{-1} = \sum_{n\geq 0}(A^{-1}B)^n A^{-1}, \quad \left|(A+B)^{-1}\right| \leq \frac{|A^{-1}|}{1-|A^{-1}B|}.$$

This section is organized as follows. At first we focus on constant matrices in finite and infinite dimensions. Then we give some results for countable non-autonomous ODE's regarding exponential dichotomy (these systems were extensively studied in [7,16,54]).

1.3.1 Finite Dimensional Case

We consider on $X = \mathbb{C}^n$ the following standard norms (c.f. [41,61])

$$|x|_p := \sqrt[p]{|x_1|^p + |x_2|^p + \cdots + |x_n|^p}, \quad |x|_\infty := \max\{|x_1|, |x_2|, \cdots, |x_n|\},$$

where $p \geq 1$ and $x = (x_1, x_2, \cdots, x_n)$. The corresponding norms on $\mathcal{B}(\mathbb{C}^n)$ are denoted by $\|\cdot\|_p$ and $\|\cdot\|_\infty$. We recall the following result [61]: If $A = (a_{ij})_{i,j=1}^n \in M^{n\times n}(\mathbb{C})$ then

$$\left.\begin{array}{c} \|A\|_1 = \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^n |a_{ji}|\right), \qquad \|A\|_2 = \sqrt{\max\limits_{1\leq i\leq n}\lambda^i_{\bar{A}^TA}}, \\ \|A\|_\infty = \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^n |a_{ij}|\right), \end{array}\right\} \tag{1.24}$$

where $\lambda^i_{\bar{A}^TA}$, $i = 1, 2, \cdots, n$ are eigenvalues of $\bar{A}^TA$.

Next, by using the Hölder inequality, for $p > 1$ we compute

$$|Ax|_p = \sqrt[p]{\sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij}x_j\right|^p} \leq \sqrt[p]{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q\right)^{p/q}\left(\sum_{j=1}^{n}|x_j|^p\right)}$$
$$= \sqrt[p]{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q\right)^{p/q}}\,|x|_p$$

for $\frac{1}{p} + \frac{1}{q} = 1$, which gives

$$\|A\|_p \leq \sqrt[p]{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q\right)^{p/q}}\,. \tag{1.25}$$

Take $\lambda \in \mathbb{C}$. In order to show the invertibility of $A_\lambda := \lambda\mathbb{I} - A$ first we suppose that

$$a_{ii} \neq \lambda, \quad \forall i = 1, 2, \cdots, n\,, \tag{1.26}$$

and then we consider the following modification of A_λ

$$\widetilde{A}_\lambda := \left(\frac{\lambda\delta_{ij} - a_{ij}}{\lambda - a_{ii}}\right)_{i,j=1}^{n}\,, \tag{1.27}$$

where

$$\delta_{ij} := \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ if } i \neq j. \end{cases}$$

Now we decompose (1.27) as follows

$$\widetilde{A}_\lambda := \mathbb{I} + B_\lambda\,, \quad B_\lambda := \left(b_{ij}^\lambda\right)_{i,j=1}^{n}\,, \quad b_{ij}^\lambda = \frac{(\delta_{ij} - 1)a_{ij}}{\lambda - a_{ii}}.$$

Note $A_\lambda = D_\lambda \widetilde{A}_\lambda$ for $D_\lambda := \operatorname{diag}(\lambda - a_{11}, \lambda - a_{22}, \cdots, \lambda - a_{nn})$. Clearly A_λ is invertible if and only if $\widetilde{A}_\lambda$ is invertible, and then $A_\lambda^{-1} = \widetilde{A}_\lambda^{-1} D_\lambda^{-1}$.

Now we have the following consequences of Lemma 1.22.

Theorem 1.23. *Suppose* (1.26) *and set* $d := \max_{1\leq i\leq n}\{|\lambda - a_{ii}|^{-1}\}$. *Then the following statements hold*

1. If

$$\eta_1 := \max_{1\le i\le n}\left\{\sum_{j=1, j\neq i}^{n} \frac{|a_{ji}|}{|\lambda - a_{jj}|}\right\} < 1\,, \tag{1.28}$$

then A_λ is invertible and $\|A_\lambda^{-1}\|_1 \le \frac{d}{1-\eta_1}$.

2. If

$$\eta_\infty := \max_{1\le i\le n}\left\{\frac{\sum_{j=1, j\neq i}^{n} |a_{ij}|}{|\lambda - a_{ii}|}\right\} < 1\,, \tag{1.29}$$

then A_λ is invertible and $\|A_\lambda^{-1}\|_\infty \le \frac{d}{1-\eta_\infty}$.

3. If

$$\tau_p := \sum_{i=1}^{n} \frac{\left(\sum_{j=1, j\neq i}^{n} |a_{ij}|^q\right)^{p/q}}{|\lambda - a_{ii}|^p} < 1\,, \tag{1.30}$$

for some $p > 1$, where $\frac{1}{p}+\frac{1}{q} = 1$, then A_λ is invertible and $\|A_\lambda^{-1}\|_p \le \frac{d}{1-\sqrt[p]{\tau_p}}$.

Proof. From (1.24) and (1.25) we get in the light of the conditions (1.28), (1.29), (1.30) that $\|B_\lambda\|_p < 1$ for any $p \in [1,\infty]$. Neumann's Lemma 1.22 implies

$$\|A_\lambda^{-1}\|_p \le \|D_\lambda^{-1}\|_p \|\widetilde{A}_\lambda^{-1}\|_p \le \frac{\|D_\lambda^{-1}\|_p}{1-\|B_\lambda\|_p}.$$

Because of $\|D_\lambda^{-1}\|_p \le d$ statements follow and the proof is finished. □

Remark 1.10. a) For $\lambda = 0$, condition (1.28) is the Hadamard classical assumption on invertibility of A (see [52]), but Hadamard ones have no estimates on the norm of A^{-1}. Further results on the invertibility of matrices are presented in [29].

b) Taking the transpose A^T we get dual results of Theorem 1.23 which here we do not present explicitly.

c) Taking opposite inequalities in the above conditions (1.28), (1.29) and (1.30), we can localize the spectrum $\sigma(A)$ by obtaining Geršgoring type sets [29,66].

d) If $\Re a_{ii} \neq 0$ for all $i = 1, 2, \cdots, n$ then using $|\lambda - a_{ii}| \ge |\Re a_{ii}|$ for all $i = 1, 2, \cdots, n$ and any $\lambda \in \mathbb{C}$ with $\Re\lambda = 0$ we see that A is hyperbolic, i.e., $\Re\sigma(A) \neq 0$, if one of the next assumptions holds

$$\left.\begin{array}{c}\max\limits_{1\le i\le n}\left\{\sum\limits_{j=1,j\ne i}^{n}\frac{|a_{ji}|}{|\Re a_{jj}|}\right\}<1\,,\quad \max\limits_{1\le i\le n}\left\{\frac{\sum_{j=1,j\ne i}^{n}|a_{ij}|}{|\Re a_{ii}|}\right\}<1\,,\\ \sum\limits_{i=1}^{n}\frac{\left(\sum_{j=1,j\ne i}^{n}|a_{ij}|^{q}\right)^{p/q}}{|\Re a_{ii}|^{p}}<1\end{array}\right\}\tag{1.31}$$

for some $p > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, A has the same type of hyperbolicity as $\operatorname{diag}(\Re a_{11}, \Re a_{22}, \cdots, \Re a_{nn})$. This follows from the fact that all matrices

$$\operatorname{diag}(\Re a_{11}, \Re a_{22}, \cdots, \Re a_{nn}) + \xi\left(A - \operatorname{diag}(\Re a_{11}, \Re a_{22}, \cdots, \Re a_{nn})\right), \quad \xi \in [0,1]$$

are hyperbolic.

1.3.2 Infinite Dimensional Case

Let us introduce for $p \in [1, \infty]$ the well-known ℓ_p Banach spaces

$$\ell_p := \left\{x = \{x_i\}_{i\in\mathbb{Z}} \,:\, x_i \in \mathbb{C},\, |x|_p < \infty\right\},$$
$$|x|_p := \sqrt[p]{\sum_{i\in\mathbb{Z}} |x_i|^p}, \quad p \in [1,\infty), \qquad |x|_\infty := \sup_{i\in\mathbb{Z}} |x_i|.$$

We deal with infinite dimensional matrices of the form

$$(Bx)_i = \sum_{j=i-s}^{i+s} b_{ij} x_j, \quad i \in \mathbb{Z}$$

for $s \in \mathbb{N}$ and a bounded sequence $\{b_{ij}\}_{i,j\in\mathbb{Z}}^{|j-i|\le s}$, where $x = \{x_i\}_{i\in\mathbb{Z}} \in \ell_p$. Then we easily derive (as we obtained (1.24))

$$|Bx|_1 = \sum_{i\in\mathbb{Z}} |(Bx)_i| \le \left(\sup_{i\in\mathbb{Z}} \sum_{j=i-s}^{i+s} |b_{ji}|\right) |x|_1,$$
$$|Bx|_\infty = \sup_{i\in\mathbb{Z}} |(Bx)_i| \le \left(\sup_{i\in\mathbb{Z}} \sum_{j=i-s}^{i+s} |b_{ij}|\right) |x|_\infty,$$
$$|Bx|_p = \sqrt[p]{\sum_{i\in\mathbb{Z}} |(Bx)_i|^p} \le \sqrt[p]{\sup_{i\in\mathbb{Z}} \sum_{k=i-s}^{i+s} \left(\sum_{j=k-s}^{k+s} |b_{kj}|^q\right)^{p/q}}\, |x|_p$$

for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. These results rely on similar tricks therefore we show only the third, the most complicated one. Let us have an element $x \in \ell_p$ such that there is $l \in \mathbb{N}$ with $x_i = 0$ if $i \in \mathbb{Z} \setminus \{-l, -l+1, \cdots, l-1, l\}$. Then

$$|Bx|_p = \sqrt[p]{\sum_{i\in\mathbb{Z}} |(Bx)_i|^p} = \sqrt[p]{\sum_{i\in\mathbb{Z}} \left| \sum_{j=i-s}^{i+s} b_{ij}x_j \right|^p} = \sqrt[p]{\sum_{i=l-s}^{l+s} \left| \sum_{j=i-s}^{i+s} b_{ij}x_j \right|^p}$$

because for $i, j \in \mathbb{Z}$ conditions $|i-l| > s$, $|i-j| \le s$ imply $|j-l| \ge |i-l| - |i-j| > 0$ hence $x_j = 0$. Further from Hölder inequality we have

$$|Bx|_p \le \sqrt[p]{\sum_{i=l-s}^{l+s} \left(\sum_{j=i-s}^{i+s} |b_{ij}|^q \right)^{\frac{p}{q}} \sum_{i=l-s}^{l+s} |x_j|^p} \le \sqrt[p]{\sup_{k\in\mathbb{Z}} \sum_{i=k-s}^{k+s} \left(\sum_{j=i-s}^{i+s} |b_{ij}|^q \right)^{\frac{p}{q}}} |x|_p.$$

Note that in the last term the coefficient before $|x|_p$ is independent on l. Now the density of the set of these x-es in ℓ_p implies the desired upper bound (some relabeling of indexes is also needed).

Let us take a matrix

$$(Ax)_i = \sum_{j=i-s}^{i+s} a_{ij}x_j, \quad i \in \mathbb{Z} \tag{1.32}$$

for $s \in \mathbb{N}$ and a bounded sequence $\{a_{ij}\}_{i,j\in\mathbb{Z}}^{|j-i|\le s}$. Take $\lambda \in \mathbb{C}$ with

$$\inf_{i\in\mathbb{Z}} |\lambda - a_{ii}| = \omega > 0, \tag{1.33}$$

introduce $A_\lambda := \mathbb{I} - \lambda A$ and also $\widetilde{A_\lambda}, B_\lambda, D_\lambda$ corresponding to sequences $\left\{\tilde{a}_{ij}^\lambda\right\}_{i,j\in\mathbb{Z}}^{|i-j|\le s}$, $\left\{b_{ij}^\lambda\right\}_{i,j\in\mathbb{Z}}^{|i-j|\le s}$ and $\left\{d_i^\lambda\right\}_{i\in\mathbb{Z}}$ (D_λ is a diagonal infinite matrix) defined as

$$\tilde{a}_{ij}^\lambda := \frac{\delta_{ij}\lambda - a_{ij}}{\lambda - a_i i}, \quad b_{ij}^\lambda := \frac{(\delta_{ij} - 1)a_{ij}}{\lambda - a_{ij}}, \quad d_i^\lambda := \lambda - a_{ii}.$$

Then $A_\lambda = D_\lambda \widetilde{A}_\lambda$, $\widetilde{A}_\lambda = \mathbb{I} + B_\lambda$ and we arrive at the following extension of Theorem 1.23 to infinite matrices.

Theorem 1.24. *Suppose* (1.33) *then the following statements hold*

1. If

$$\eta_1 := \sup_{i\in\mathbb{Z}} \sum_{j=i-s, j\ne i}^{j=i+s} \frac{|a_{ji}|}{|\lambda - a_{jj}|} < 1\,, \tag{1.34}$$

then A_λ is invertible in ℓ_1 and $\|A_\lambda^{-1}\|_1 \leq (\omega(1-\eta_1))^{-1}$.

2. If

$$\eta_\infty := \sup_{i\in\mathbb{Z}} \sum_{j=i-s, j\neq i}^{i+s} \frac{|a_{ij}|}{|\lambda - a_{ii}|} < 1\,, \tag{1.35}$$

then A_λ is invertible in ℓ_∞ and $\|A_\lambda^{-1}\|_\infty \leq (\omega(1-\eta_\infty))^{-1}$.

3. If

$$\tau_p := \sup_{i\in\mathbb{Z}} \sum_{k=i-s}^{i+s} \frac{\left(\sum_{j=k-s, j\neq k}^{k+s} |a_{kj}|^q\right)^{p/q}}{|\lambda - a_{kk}|^p} < 1\,, \tag{1.36}$$

for some $p \in (1,\infty)$, where $\frac{1}{p}+\frac{1}{q}=1$, then A_λ is invertible in ℓ_p and

$$\|A_\lambda^{-1}\|_p \leq \left(\omega(1-\sqrt[p]{\tau_p})\right)^{-1}.$$

Proof. The idea is the same as the proof of Theorem 1.23. Formally the only difference is that $\|D_\lambda^{-1}\|_p \leq 1/\omega$. □

Remark 1.11. Assuming $\inf_{i\in\mathbb{Z}} |\Re a_{ii}| > 0$ we infer that A is hyperbolic

$$\begin{gathered}
\text{in } \ell_1 \text{ if} \sup_{i\in\mathbb{Z}} \sum_{j=i-s, j\neq i}^{i+s} \frac{|a_{ji}|}{|\Re a_{jj}|} < 1\,, \quad \text{in } \ell_\infty \text{ if} \sup_{i\in\mathbb{Z}} \frac{\sum_{j=i-s, j\neq i}^{i+s} |a_{ij}|}{|\Re a_{ii}|} < 1\,, \\
\text{in } \ell_p \text{ if} \sup_{i\in\mathbb{Z}} \sum_{k=i-s}^{i+s} \frac{\left(\sum_{j=k-s, j\neq k}^{k+s} |a_{kj}|^q\right)^{p/q}}{|\Re a_{kk}|^p} < 1\,,
\end{gathered} \tag{1.37}$$

for some $p > 1$ where $\frac{1}{p}+\frac{1}{q}=1$. It is enough to use Theorem 1.24 for $\lambda \in \mathbb{C}$ such that $\Re\lambda = 0$ and note that then $|\lambda - a_{ii}| \geq |\Re a_{ii}|$. Moreover, A has the same type of hyperbolicity as $\operatorname{diag}(\Re a_{ii})_{i\in\mathbb{Z}}$.

Note that these results are the direct consequences of the previous subsection. More sophisticated results are presented in [62, 65] on spectra of infinite matrices.

1.3.3 Periodic ODE's

Consider a first order T-periodic ODE

$$\dot{x} = A(t)x \tag{1.38}$$

with

$$(A(t)x)_i = \sum_{j=i-s}^{i+s} a_{ij}(t)x_j, \qquad i \in \mathbb{Z},$$

for $s \in \mathbb{N}$ and a uniformly bounded sequence $\{a_{ij}(t)\}_{i,j\in\mathbb{Z}}^{|j-i|\le s}$ of T-periodic continuous functions. First we suppose that

$$\lim_{i\to\pm\infty} a_{i,i+k}(t) = a_k^{\pm}(t), \qquad \forall k \in \{-s,-s+1,\cdots,s-1,s\} \tag{1.39}$$

uniformly on $[0,T]$. Then we set

$$(A_\infty(t)x)_i = \begin{cases} \sum_{j=i-s}^{i+s} a_{j-i}^{+}(t)x_j, & i \ge 0, \\ \sum_{j=i-s}^{i+s} a_{j-i}^{-}(t)x_j, & i < 0, \end{cases}$$

It is an easy exercise to verify that $C(t) := A(t) - A_\infty(t)$ are compact in any ℓ_p, $p \in [1,\infty]$ for all $t \in [0,T]$ ($T \in \mathcal{B}(X,Y)$ is compact if the closure $\overline{T(M)}$ is compact in Y for arbitrary bounded $M \subset X$). The fundamental matrix solution $\mathcal{X}(t)$ of (1.38) has the form

$$\mathcal{X}(t) = \mathcal{X}_\infty(t) + \int_0^t \mathcal{X}_\infty(t)\big(\mathcal{X}_\infty(z)\big)^{-1}C(z)\mathcal{X}(z)\mathrm{d}z,$$

where $\mathcal{X}_\infty$ is the fundamental matrix solution of $\dot{x} = A_\infty(t)x$ (this is a consequence of (1.4), because $Y(t) = \mathcal{X}(t)$ is a matrix solution of the linear equation $\dot{Y} = A_\infty(t)Y + C(t)\mathcal{X}(t)$). Hence $\mathcal{X}(T) - \mathcal{X}_\infty(T)$ is compact and so $\sigma_{ess}(\mathcal{X}(T)) = \sigma_{ess}(\mathcal{X}_\infty(T))$ (the essential spectrum $\sigma_{ess}(T)$ of $T \in \mathcal{B}(X)$ is not uniquely defined in the literature, one can find various definitions [20, p. 40], we follow the most general one

$$\begin{aligned} \sigma_{ess}(T) :=& \{\lambda \in \mathbb{C} \,:\, \lambda\mathbb{I} - T \text{ is not Fredholm }\} \\ =& \{\lambda \in \mathbb{C} \,:\, \dim(\mathrm{Ker}(\lambda\mathbb{I} - T) = \infty \text{ or } \mathrm{codim}(R(\lambda\mathbb{I} - T)) = \infty\}\,. \end{aligned}$$

The fundamental fact is that $\sigma_{ess}(\cdot)$ is invariant under the compact perturbations, that is $\sigma_{ess}(T + C) = \sigma(T)$ for any compact $C \in \mathcal{B}(X)$, c.f. [20, Theorem 4.1, p. 40]). Then for instance, if $\mathcal{X}_\infty(t) = 0$ then $\mathcal{X}(T)$ is a compact perturbation of $\mathbb{I}$.

We recall [15, Theorem 2.1, p. 203] that (1.38) has an exponential dichotomy on $\mathbb{R}$ if and only if $\sigma(X(T)) \cap S^1 = \emptyset$ for the unit circle S^1 in $\mathbb{C}$. This is equivalent to say that the inhomogeneous system

$$\dot{x} = A(t)x + h(t) \tag{1.40}$$

has a unique bounded solution on $\mathbb{R}$ for any bounded continuous $h \in C_b(\mathbb{R}, X)$ (here X is a complex Banach space, namely one of the ℓ_p spaces for $p \in [1, \infty]$). Now we rewrite (1.40) as a system

$$\dot{x}_i = a_{ii}(t)x_i + \sum_{j=i-s, j\neq i}^{i+s} a_{ij}(t)x_j + h_i(t), \quad i \in \mathbb{Z} \tag{1.41}$$

for $h(t) = \{h_i(t)\}_{i\in\mathbb{Z}}$. We suppose

$$\omega := \inf_{i\in\mathbb{Z}, t\in\mathbb{R}} |\Re a_{ii}(t)| > 0. \tag{1.42}$$

We want to find criteria that (1.41) has a unique bounded solution on $\mathbb{R}$. For this purpose, we rewrite it as

$$x_i(t) = \int_{a_i\infty}^{t} \mathrm{e}^{A_i(t,z)} \sum_{j=i-s, j\neq i}^{i+s} a_{ij}(z)x_j(z)\mathrm{d}z + \int_{a_i\infty}^{t} \mathrm{e}^{A_i(t,z)} h_i(z)\mathrm{d}z, \quad i \in \mathbb{Z},$$

where $a_i := \operatorname{sign} \Re a_{ii}(t)$ and $A_i(t, z) := \int_z^t a_{ii}(u)\mathrm{d}u$. Note $|\Re A_i(t, z)| \geq \omega|t - z|$.

Then for $x, h \in C_b(\mathbb{R}, \ell_\infty)$ we derive

$$\begin{aligned}
|x_i(t)| \leq & \sup_{i\in\mathbb{Z}, z\in\mathbb{R}} \sum_{j=i-s, j\neq i}^{i+s} \frac{|a_{ij}(z)|}{|\Re a_{ii}(z)|} \left(-a_i \int_{a_i\infty}^{t} |\Re a_{ii}(z)| \, \mathrm{e}^{\Re A_i(t,z)}\mathrm{d}z\right) |x|_\infty \\
& + |h|_\infty(-a_i) \int_{a_i\infty}^{t} \mathrm{e}^{a_i\omega(t-z)}\mathrm{d}z \\
= & \sup_{i\in\mathbb{Z}, z\in\mathbb{R}} \sum_{j=i-s, j\neq i}^{i+s} \frac{|a_{ij}(z)|}{|\Re a_{ii}(z)|} |x|_\infty + \frac{|h|_\infty}{\omega}.
\end{aligned}$$

Consequently, if

$$\sup_{i\in\mathbb{Z}, z\in\mathbb{R}} \sum_{j=i-s, j\neq i}^{i+s} \frac{|a_{ij}(z)|}{|\Re a_{ii}(z)|} < 1$$

then (1.40) has a unique solution $x \in C_b(\mathbb{R}, \ell_\infty)$ for any $h \in C_b(\mathbb{R}, \ell_\infty)$, and thus (1.38) has an exponential dichotomy on ℓ_∞.

Similarly for $x, h \in C_b(\mathbb{R}, \ell_1)$ we derive

$$
\begin{aligned}
|x(t)|_1 = \sum_{i\in\mathbb{Z}} |x_i(t)| \le \sum_{i\in\mathbb{Z}} -a_i \int_{a_i\infty}^{t} \mathrm{e}^{a_i\omega(t-z)} \sum_{j=i-s, j\neq i}^{i+s} |a_{ij}(z)||x_j(z)|\mathrm{d}z \\
+ \sum_{i\in\mathbb{Z}} -a_i \int_{a_i\infty}^{t} \mathrm{e}^{a_i\omega(t-z)} |h_i(z)|\mathrm{d}z \\
= \sum_{j\in\mathbb{Z}} \sum_{i=j-s, i\neq j}^{j+s} -a_i \int_{a_i\infty}^{t} \mathrm{e}^{a_i\omega(t-z)} |a_{ij}(z)||x_j(z)|\mathrm{d}z + \sum_{i\in\mathbb{Z}} -a_i \int_{a_i\infty}^{t} \mathrm{e}^{a_i\omega(t-z)} |h_i(z)|\mathrm{d}z \\
= \sum_{j\in\mathbb{Z}} \left(\sum_{0<|i-j|\le s}^{a_i=-1} \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-z)} |a_{ij}(z)||x_j(z)|\mathrm{d}z + \sum_{0<|i-j|\le s}^{a_i=1} \int_{t}^{\infty} \mathrm{e}^{\omega(t-z)} |a_{ij}(z)||x_j(z)|\mathrm{d}z \right) \\
+ \sum_{i\in\mathbb{Z}}^{a_i=-1} \int_{-\infty}^{t} \mathrm{e}^{-\omega(t-z)} |h_i(z)|\mathrm{d}z + \sum_{i\in\mathbb{Z}}^{a_i=1} \int_{t}^{\infty} \mathrm{e}^{\omega(t-z)} |h_i(z)|\mathrm{d}z \\
\le \frac{\sup_{j\in\mathbb{Z}} \left(\sup_{z\in\mathbb{R}} \sum_{0<|i-j|\le s}^{a_i=-1} |a_{ij}(z)| + \sup_{z\in\mathbb{R}} \sum_{0<|i-j|\le s}^{a_i=1} |a_{ij}(z)| \right)}{\omega} |x|_\infty + \frac{2|h|_\infty}{\omega},
\end{aligned}
$$

which implies

$$
|x|_\infty \le \frac{\sup_{j\in\mathbb{Z}} \left(\sup_{z\in\mathbb{R}} \sum_{0<|i-j|\le s}^{a_i=-1} |a_{ij}(z)| + \sup_{z\in\mathbb{R}} \sum_{0<|i-j|\le s}^{a_i=1} |a_{ij}(z)| \right)}{\omega} |x|_\infty + \frac{2|h|_\infty}{\omega}.
$$

Consequently, if

$$
\sup_{j\in\mathbb{Z}} \left(\sup_{z\in\mathbb{R}} \sum_{0<|i-j|\le s}^{a_i=-1} |a_{ij}(z)| + \sup_{z\in\mathbb{R}} \sum_{0<|i-j|\le s}^{a_i=1} |a_{ij}(z)| \right) < \omega
$$

then (1.40) has a unique solution $x \in C_b(\mathbb{R}, \ell_1)$ for any $h \in C_b(\mathbb{R}, \ell_1)$, and thus (1.38) has an exponential dichotomy on ℓ_1.

Finally for $x, h \in C_b(\mathbb{R}, \ell_p)$, $p \in (1, \infty)$ we derive

$$|x(t)|_p = \sqrt[p]{\sum_{i\in\mathbb{Z}} \left| -a_i \int_{a_i\infty}^t \mathrm{e}^{A_i(t,s)} \sum_{j=i-s, j\neq i}^{i+s} |a_{ij}(z)| x_j(z) \mathrm{d}z - a_i \int_{a_i\infty}^t \mathrm{e}^{A_i(t,z)} h_i(z) \mathrm{d}z \right|^p}$$

$$\leq \sqrt[p]{\sum_{i\in\mathbb{Z}} \left| \int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)} \sum_{j=i-s, j\neq i}^{i+s} |a_{ij}(z)||x_j(z)| \mathrm{d}z \right|^p}$$

$$+ \sqrt[p]{\sum_{i\in\mathbb{Z}} \left| \int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)} |h_i(z)| \mathrm{d}z \right|^p}$$

$$\leq \sqrt[p]{\sum_{i\in\mathbb{Z}} \left| \int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)} \sqrt[q]{\sum_{j=i-s, j\neq i}^{i+s} |a_{ij}(z)|^q} \sqrt[p]{\sum_{j=i-s, j\neq i}^{i+s} |x_j(z)|^p \mathrm{d}z} \right|^p}$$

$$+ \sqrt[p]{\sum_{i\in\mathbb{Z}} \left| \int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)} \mathrm{d}z \right|^{p/q} \left| \int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)} |h_i(z)|^p \mathrm{d}z \right|}$$

$$\leq \left\{ \sum_{i\in\mathbb{Z}} \left[\left| \int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)} \mathrm{d}z \right|^{p/q} \cdot \right.\right.$$

$$\left.\left. \left| \int_{a_i\infty}^t \mathrm{e}^{a_i\omega(t-z)} \left(\sum_{j=i-s, j\neq i}^{i+s} |a_{ij}(z)|^q \right)^{p/q} \left(\sum_{j=i-s, j\neq i}^{i+s} |x_j(z)|^p \right) \mathrm{d}z \right| \right] \right\}^{1/p}$$

$$+ \sqrt[q]{\frac{1}{\omega}} \sqrt[p]{\sum_{i\in\mathbb{Z}}^{a_i=-1} \int_{-\infty}^t \mathrm{e}^{-\omega(t-z)} |h_i(z)|^p \mathrm{d}z + \sum_{i\in\mathbb{Z}}^{a_i=1} \int_t^\infty \mathrm{e}^{\omega(t-z)} |h_i(z)|^p \mathrm{d}z}$$

$$\leq \sqrt[q]{\frac{1}{\omega}} \left(\sum_{j\in\mathbb{Z}} \int_{-\infty}^t \mathrm{e}^{-\omega(t-z)} \sum_{0<|i-j|\leq s}^{a_i=-1} \left(\sum_{k=i-s, k\neq i}^{i+s} |a_{ik}(z)|^q \right)^{p/q} |x_j(z)|^p \mathrm{d}z \right.$$

$$\left. + \sum_{j\in\mathbb{Z}} \int_t^\infty \mathrm{e}^{\omega(t-z)} \sum_{0<|i-j|\leq s}^{a_i=1} \left(\sum_{k=i-s, k\neq i}^{i+s} |a_{ik}(z)|^q \right)^{p/q} |x_j(z)|^p \mathrm{d}z \right)^{1/p} + \frac{\sqrt[p]{2}}{\omega} |h|_\infty$$

$$\leq \frac{1}{\omega} \left(\sup_{j\in\mathbb{Z}, z\in\mathbb{R}} \sum_{0<|i-j|\leq s}^{a_i=-1} \left(\sum_{k=i-s, k\neq i}^{i+s} |a_{ik}(z)|^q \right)^{p/q} \right.$$

$$\left. + \sup_{j\in\mathbb{Z}, z\in\mathbb{R}} \sum_{0<|i-j|\leq s}^{a_i=1} \left(\sum_{k=i-s, k\neq i}^{i+s} |a_{ik}(z)|^q \right)^{p/q} \right)^{1/p} |x|_\infty + \frac{\sqrt[p]{2}}{\omega} |h|_\infty.$$

Consequently, if

$$\sup_{j\in\mathbb{Z}, z\in\mathbb{R}} \sum_{0<|i-j|\leq s}^{a_i=-1} \left(\sum_{k=i-s, k\neq i}^{i+s} |a_{ik}(z)|^q \right)^{p/q} + \sup_{j\in\mathbb{Z}, z\in\mathbb{R}} \sum_{0<|i-j|\leq s}^{a_i=1} \left(\sum_{k=i-s, k\neq i}^{i+s} |a_{ik}(z)|^q \right)^{p/q} < \omega^p$$

then (1.40) has a unique solution $x \in C_b(\mathbb{R}, \ell_p)$ for any $h \in C_b(\mathbb{R}, \ell_p)$, and thus (1.38) has an exponential dichotomy on ℓ_p (note that in the above computations $|x|_\infty = \sup_{t\in\mathbb{R}} |x(t)|_p$ strongly depends on $p \in [1, \infty]$).

Finally we consider a second-order ODE

$$\ddot{x} = A(t)x \tag{1.43}$$

with

$$(A(t)x)_i = \sum_{j=i-s}^{i+s} a_{ij}(t)x_j, \qquad i \in \mathbb{Z}$$

for $s \in \mathbb{N}$ and a uniformly bounded sequence $\{a_{ij}(t)\}_{i,j\in\mathbb{Z}}^{|j-i|\le s}$ of T-periodic continuous functions. We say that this equation possesses an exponential dichotomy on $\mathbb{R}$ – generally on a suitable Banach space X – if the corresponding first order system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I}_X \\ A(t) & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

is exponentially dichotomous on $X \times X$. By following [15, Theorem 5.1 p. 32, Theorem 2.4 p. 208] we know that (1.43) on the Hilbert space $X = \ell_2$ possess an exponential dichotomy on $\mathbb{R}$ if $\sigma(R(t)) > 0$ for any $t \in \mathbb{R}$ where $R(t) := \Re A(t) = \frac{1}{2}(A(t) + A^*(t))$, $A^*(t)$ is the adjoint of $A(t)$ (defined through $(Ax, y) = (x, A^*y)$) and $\sigma(R(t)) > 0$ means $\Re\lambda > 0, \forall\lambda \in \sigma(R(t)), t \in \mathbb{R}$. It is an easy computation to show

$$(R(t)x)_i = \sum_{j=i-s}^{i+s} r_{ij}x_j,\ i \in \mathbb{Z},$$
$$r_{ij}(t) := \frac{1}{2}(a_{ij}(t) + \overline{a}_{ji}(t)), \qquad t \in \mathbb{R},\ i, j \in \mathbb{Z},\ |i-j| \le s.$$

Let us suppose in the light of (1.33) that

$$\inf_{i\in\mathbb{Z}} r_{ii} = \inf_{i\in\mathbb{Z}} \Re a_{ii} > 0.$$

Using Theorem 1.24 (the third part with $p = 2$) for $\lambda \in \mathbb{C}, \Re\lambda \le 0$ (noting that

$|\lambda - r_{ii}| \geq \Re r_{ii} = \Re a_{ii}$) we get that if

$$\sup_{i\in\mathbb{Z}} \sum_{k=i-s}^{i+s} \frac{\sum_{j=k-s, j\neq k}^{k+s} |(a_{kj}(t) + \overline{a}_{jk}(t))/2|^2}{\Re a_{kk}(t)^2} < 1$$

then $\sigma(R(t)) > 0, t \in \mathbb{R}$ and hence (1.43) is exponentially dichotomous on $\mathbb{R}$.

Chapter 2

Discretization of Poincaré map

2.1 General Settings and Tools

Assumptions made here are going to be valid for the whole chapter. Let us have $f \in C^3(\mathbb{R}^N)$, $N \in \mathbb{N} \setminus \{1\}$ such that

$$\varphi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \text{ is the global flow of } \dot{x} = f(x).$$

For a numerical scheme $\psi : [0, h_0] \times \mathbb{R}^N \to \mathbb{R}^N$, $h_0 \in (0,1)$ suppose for some $p \in \mathbb{N}$ that

$$\psi(h, x) = \varphi(h, x) + \Upsilon(h, x) h^{p+1}. \tag{2.1}$$

Assume again $\psi, \Upsilon \in C^3([0, h_0] \times \mathbb{R}^N, \mathbb{R}^N)$. Some technical reasons cause that we are forced to assume also $p \geq 2$ (see below Remark 2.2 for more details).

Let $\gamma(s) := \varphi(s, \xi_0)$ be a 1-periodic solution for fixed $\xi_0 \in \mathbb{R}^N$. Then there is a system $\{e_i(s)\}_{i=1}^{N-1}$ of vectors in $\mathbb{R}^N$ for any $s \in \mathbb{R}$ such that

$$\left.\begin{aligned} e_i \in C^3(\mathbb{R}, \mathbb{R}^N), &\quad e_i(s+1) = e_i(s), \\ \langle e_i(s), e_j(s) \rangle = \delta_{ij}, &\quad \langle e_i(s), f(\gamma(s)) \rangle = 0, \end{aligned}\right\} \tag{2.2}$$

where $i, j \in \{1, \cdots, N-1\}$, δ_{ij} is a Kronecker's delta and $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product. Introduce an $N \times (N-1)$ matrix $E(s) = [e_1, \cdots, e_{N-1}]$ (i-th column is e_i, $i = 1, \cdots, N-1$). Let us set also a tubular coordinate function $\xi(s, c) := \gamma(s) + E(s)c$ for $s \in \mathbb{R}, c \in \mathbb{R}^{N-1}$. For standard euclidian norm

$|c|_2 := \sqrt{\langle c, c\rangle}$ note that $|E(s)c|_2 = |c|_2, c \in \mathbb{R}^{N-1}$. For $\delta > 0$ introduce the notation $B^{\delta}_{N-1} := \left\{c \in \mathbb{R}^{N-1} \,:\, |c|_2 < \delta\right\}$. Using the implicit function theorem finite number of times we get that there is a $\delta_{\text{tr}} > 0$ such that

$$\xi : [0,1) \times B^{\delta_{\text{tr}}}_{N-1} \to \mathbb{R}^N \text{ is a } C^3\text{–transformation,}$$

in other words $\xi|_{[0,1)\times B^{\delta_{\text{tr}}}_{N-1}}$ is a C^3–diffeomorphism between its domain and range (cf. the moving orthonormal system along γ in [32, Chapter VI.I., p. 214-219]) . For values

$$h \in [0, h_0],\ s \in \mathbb{R},\ c \in \mathbb{R}^{N-1},\ \Delta \in [0, h_0],$$
$$X := (x^1, x^2, \cdots, x^{m-1}) \in \mathbb{R}^{N(m-1)},\ x^i \in \mathbb{R}^N,\ m \in \mathbb{N},\ m \geq 4,$$

define the following useful functions

$$\begin{aligned}
F_m(h, s, c, X, \Delta) :=& (G_m(h, s, c, X), H_m(h, s, c, X, \Delta)),\\
G_m(h, s, c, X) :=& \big(\psi(h, \xi(s, c)) - x^1, \psi(h, x^1) - x^2, \psi(h, x^2) - x^3,\\
& \cdots, \psi(h, x^{m-2}) - x^{m-1}\big),\\
H_m(h, s, c, X, \Delta) :=& \left\langle \psi\left(\Delta, x^{m-1}\right) - \gamma(s), f(\gamma(s))\right\rangle.\\
\bar{X}_m :=& \bar{X}_m(h, s, c) := \left(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^{m-1}\right),\\
\bar{x}^j :=& \bar{x}^j(h, s, c) := \varphi(jh, \xi(s, c)),\ j = 1, 2, \cdots, m-1.
\end{aligned}$$

Further let B be a compact set such that $\gamma(\mathbb{R})$ is contained in the interior of B. Hence there is a constant $R > 0$ such that

$$\left\{x \in \mathbb{R}^N \,:\, \min_{s\in\mathbb{R}}\{|x - \gamma(s)|\} \leq R\right\} \subset B. \tag{2.3}$$

We mean by $|\cdot|$ the standard maximum norm $|v| := \max\{|v_i| \,:\, i = 1, \cdots, l\}$ for $v \in \mathbb{R}^l, l \in \mathbb{N}$. Notation$|\cdot|$ is used also for linear operators $A : \mathbb{R}^{l_1} \to \mathbb{R}^{l_2}$ defined as $|A| := \max_{v\in\mathbb{R}^{l_1},|v|=1} |Av|$. An open ball in a Banach space X will be denoted as $B(x, \varrho) := \{y \in X \,:\, |y - x| < \varrho\}$ for any $x \in X$ and $\varrho > 0$.

Several times we will use Lemma 1.22 from the previous chapter. Our central tool will be the following lemma, because of it importance and for the sake of completeness we give also a short proof.

Lemma 2.1 (Newton–Kantorovich method)**.** *Let us have Banach spaces X, Y, Z and open nonempty sets $U \subset X, V \subset Y$. Let $\bar{y} : U \to V$ be any function such that*

$$\overline{B(\bar{y}(x), \varrho)} \subset V \text{ for every } x \in U \text{ and for some } \varrho > 0.$$

Let us have a function $F \in C^r(U \times V, Z)$ for $r \geq 1$. Suppose that

$$D_y F(x, \bar{y}(x))^{-1} \in \mathcal{B}(Z, Y),$$
$$|F(x, \bar{y}(x))| \leq \alpha, \quad |D_y F(x, \bar{y}(x))^{-1}| \leq \beta$$

for every $x \in U$ and for some $\alpha, \beta > 0$. Let

$$|D_y F(x, y_1) - D_y F(x, y_2)| \leq l|y_1 - y_2|, \quad x \in U, y_1, y_2 \in \overline{B(\bar{y}(x), \varrho)} \tag{2.4}$$

hold for some $l \geq 0$. For constants $\alpha, \beta, l, \varrho$ finally suppose

$$\beta l \varrho < 1, \tag{2.5}$$
$$\alpha\beta < \varrho(1 - \beta l \varrho). \tag{2.6}$$

Then there is a unique function $\mathrm{y} : U \to V$ such that

$$|\mathrm{y}(x) - \bar{y}(x)|_Y \leq \varrho \text{ and } F(x, \mathrm{y}(x)) = 0 \text{ for all } x \in U.$$

Moreover

$$|\mathrm{y}(x) - \bar{y}(x)| < \varrho, \quad D_y F(x, \mathrm{y}(x))^{-1} \in \mathcal{B}(Z, Y)$$

for all $x \in U$ with an estimate

$$\left|D_y F(x, \mathrm{y}(x))^{-1}\right| \leq \frac{\beta}{1 - \beta l \varrho}.$$

We get also $\mathrm{y} \in C^r(U, V)$ if we additionally assume the continuity of $\bar{y}$.

Proof. We transform the task to the fixed point problem of the mapping

$$G(x,y) := y - [D_yF(x,\bar{y}(x))]^{-1}F(x,y).$$

Choose an arbitrary $x \in U$. For $y_1, y_2 \in \overline{B(\bar{y}(x),\varrho)}$ we have

$$\begin{aligned}
|G(x,y_1) - G(x,y_2)| &= \left|y_1 - y_2 - D_yF(x,\bar{y}(x))^{-1}(F(x,y_1) - F(x,y_2))\right| \\
&\le \beta \left| \int_0^1 \left[D_yF(x,\bar{y}(x)) - D_yF(x, y_2 + s(y_1-y_2))\right](y_1-y_2)\mathrm{d}s \right| \\
&\le \beta l \int_0^1 |y_2 + s(y_1 - y_2)|\,|y_1 - y_2|\mathrm{d}s \le \beta l \varrho |y_1 - y_2|
\end{aligned}$$

where we used $y_2 + s(y_1 - y_2) \in \overline{B(\bar{y}(x),\varrho)}$ which is caused by the convexity of the closed ball $\overline{B(\bar{y}(x),\varrho)}$.

On the other hand $|G(x,\bar{y}(x)) - \bar{y}(x)| \le \beta\alpha < \varrho(1-\beta l\varrho)$ (cf. (2.6)) implies for any $y \in \overline{B(\bar{y}(x),\varrho)}$ that

$$\begin{aligned}
|G(x,y) - \bar{y}(x)| \le& |G(x,y) - G(x,\bar{y}(x))| + |G(x,\bar{y}(x)) - \bar{y}(x)| \\
<& \beta l\varrho|y - \bar{y}(x)| + \varrho(1-\beta l\varrho) \le \beta l \varrho^2 + \varrho(1-\beta l\varrho) = \varrho
\end{aligned}$$

therefore $G(x,\cdot) : \overline{B(\bar{y}(x),\varrho)} \to B(\bar{y}(x),\varrho) \subset \overline{B(\bar{y}(x),\varrho)}$ and it is a contraction (from (2.5)). Banach's Theorem yields a unique fixed point $\mathrm{y}(x) \in \overline{B(\bar{y}(x),\varrho)}$ of this mapping which lies in $B(\bar{y}(x),\varrho)$.

Now because of

$$\begin{aligned}
D_yF(x,\mathrm{y}(x)) &= D_yF(x,\bar{y}(x)) + D_yF(x,\mathrm{y}(x)) - D_yF(x,\bar{y}(x)) \\
&= D_yF(x,\bar{y}(x))\left[\mathbb{I} + D_yF(x,\bar{y}(x))^{-1}\left(D_yF(x,\mathrm{y}(x)) - D_yF(x,\bar{y}(x))\right)\right]
\end{aligned}$$

and

$$\left|D_yF(x,\bar{y}(x))^{-1}\left(D_yF(x,\mathrm{y}(x)) - D_yF(x,\bar{y}(x))\right)\right| \le \beta l \varrho < 1$$

we get from Lemma 1.22 that $D_yF(x,\mathrm{y}(x))$ is invertible with

$$\begin{aligned}\left|D_yF(x,\mathrm{y}(x))^{-1}\right| \leq& \frac{|D_yF(x,\bar{y}(x))^{-1}|}{1-|D_yF(x,\bar{y}(x))^{-1}(D_yF(x,\mathrm{y}(x))-D_yF(x,\bar{y}(x))|}\\ \leq& \frac{\beta}{1-\beta l\varrho}.\end{aligned}$$

Now we show C^r–smoothness. Choose any $x_0 \in U$ and let $y_0 := \mathrm{y}(x_0)$. From the results above we have $F(x_0,y_0) = 0$ and that $D_yF(x_0,y_0)$ is continuously invertible. Implicit Function Theorem yields a unique function $y^\star \in C^r(U',V')$ such that $F(x,y) = 0$ holds for $(x,y) \in U' \times V'$ if and only if $y = y^\star(x)$. Here U', V' are sufficiently small open sets with properties

$$x_0 \in U' \subset U, \quad y_0 \in V' \subset B(\bar{y}(x_0),\varrho).$$

Next

$$|y^\star(x) - \bar{y}(x)| \leq |y^\star(x) - y_0| + |y_0 - \bar{y}(x_0)| + |\bar{y}(x_0) - \bar{y}(x)|.$$

Here the second term is $< \varrho$ and other two terms are arbitrary small if x is sufficiently close to x_0 (because of the continuity of $y^\star$ and $\bar{y}$ at x_0). Therefore we have an open set U'' such that $x_0 \in U'' \subset U'$ and for which $|y^\star(x) - \bar{y}(x)| < \varrho$ for every $x \in U''$. The uniqueness of the first part of this proof ensures that $\mathrm{y} = y^\star$ on U'', so $\mathrm{y}|_{U''} \in C^r(U'',V)$. Because x_0 was chosen arbitrarily in U we get also $\mathrm{y} \in C^r(U,V)$ and the proof is finished. □

2.2 Numerical Poincaré Map

At first as a basis for further investigation we state a lemma about the global Poincaré map for the continuous dynamical systems.

Lemma 2.2 (Poincaré's time return map)**.** *There is an $\varepsilon^\star \in (0, 1/2)$ such that for every $\varepsilon \in (0, \varepsilon^\star]$ there is $\delta_{\text{re}} = \delta_{\text{re}}(\varepsilon) \in (0, \delta_{\text{tr}}]$ and a C^3–function*

$$\tau : \mathbb{R} \times B_{N-1}^{\delta_{\text{re}}(\varepsilon)} \to (1-\varepsilon, 1+\varepsilon)$$

such that for $t \in (1-\varepsilon, 1+\varepsilon)$, $s \in \mathbb{R}$ and $c \in B_{N-1}^{\delta_{\text{re}}(\varepsilon)}$ we have

$$z(t, s, c) = 0 \text{ for } z(t, s, c) := \langle \varphi(t, \xi(s, c)) - \gamma(s), f(\gamma(s)) \rangle \tag{2.7}$$

if and only if $t = \tau(s, c)$. In addition $\tau(s+1, \cdot) = \tau(s, \cdot)$, $s \in \mathbb{R}$.

Proof. The C^3–smoothness of $z : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R}$ is straightforward. It is easy to see that

$$z(1, s, 0) = 0 \text{ and } D_t z(t, s, c)|_{t=1, c=0} = |f(\gamma(s))|_2^2 \neq 0.$$

From the Implicit Function Theorem we get for all $s' \in [0, 1]$ numbers $\delta(s') > 0, \eta(s') > 0, \varepsilon(s') \in (0, 1/2)$ and C^3–smooth implicit functions

$$\tau^{s'} : (s' - \eta(s'), s' + \eta(s')) \times B_{N-1}^{\delta(s')} \to (1 - \varepsilon(s'), 1 + \varepsilon(s'))$$

determined uniquely by the equation (2.7) for

$$(t, s, c) \in (1 - \varepsilon(s'), 1 + \varepsilon(s')) \times (s' - \eta(s'), s' + \eta(s')) \times B_{N-1}^{\delta(s')}.$$

Now $\bigcup_{s' \in [0,1]} \big(s' - \eta(s')/2, s' + \eta(s')/2\big) \supset [0, 1]$ so we can choose finite number of elements $0 \leq s_1 \leq \cdots \leq s_k \leq 1$ such that $\bigcup_{i=1}^{k} \big(s_i - \eta(s_i)/2, s_i + \eta(s_i)/2\big) \supset [0, 1]$. Introduce

$$\delta := \min\{\delta_{\text{tr}}, \min_{i=1,\cdots,k} \{\delta(s_i)\}\} \text{ and } \varepsilon^\star := \min_{i=1,\cdots,k} \{\varepsilon(s_i)\}.$$

Now $\tau^{s_i}(s, 0) = 1$ together with uniform continuity of τ^{s_i} on $[s_i - \eta(s_i)/2, s_i + \eta(s_i)/2] \times \overline{B_{N-1}^{\delta/2}}$ implies that for every $\varepsilon \in (0, \varepsilon^\star]$ there is a $\delta_{\text{re}} = \delta_{\text{re}}(\varepsilon) \in (0, \delta/2]$

such that

$$\tau^{s_i}(s,c) \in (1-\varepsilon, 1+\varepsilon)$$

for all $i = 1, \cdots, k$, $s \in \big(s_i - \eta(s_i)/2, s_i + \eta(s_i)/2\big)$ and $c \in B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon)}$. Therefore $\tau : [0,1] \times B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon)} \to (1-\varepsilon, 1+\varepsilon)$ can be defined by functions τ^{s_i} naturally as follows, for $s \in [0,1]$ choose any s_i such that $s \in \big(s_i - \eta(s_i)/2, s_i + \eta(s_i)/2\big)$ then set $\tau(s, \cdot) := \tau^{s_i}(s, \cdot)$ (equality of τ^{s_i} and τ^{s_j} on the intersection of their domains comes from $\tau^{s_i}(s,0) = 1 = \tau^{s_j}(s,0)$ and the Implicit Function Theorem – so τ is well-defined).

Because the determining equation (2.7) is $1-$periodic in s we can easily extend τ to become a function $\mathbb{R} \times B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon)} \to (1-\varepsilon, 1+\varepsilon)$ which is 1-periodic in the first variable by the identity $\tau(s+k, c) := \tau(s,c), k \in \mathbb{Z}, s \in [0,1]$. The proof is complete. □

In this context the usual Poincaré map is defined as

$$\mathcal{P}(s,c) := \varphi(\tau(s,c), \xi(s,c)).$$

Further for admissible values of (h,s,c) using τ from the above lemma introduce

$$\bar{\Delta}_m := \bar{\Delta}_m(h,s,c) := \tau(s,c) - (m-1)h.$$

To get the exact meaning of $\mathcal{P}_m$ mentioned informally in the introduction we have to solve the equation $F_m(h,s,c,X,\Delta) = 0$ near $(\bar{X}, \bar{\Delta})$. Here comes the first application of Lemma 2.1. Before this let us introduce some technicalities, at first the following *positive* constants

$$
\left.
\begin{aligned}
&\mathrm{C}_\Upsilon \geq \max_{\substack{h\in[0,h_0],\, x\in B,\\ k\in\{0,1,2,3\}}} \left\{|D^{[k]}\Upsilon(h,x)|\right\},\\
&\mathrm{C}_\varphi \geq \max\Bigg\{ \max_{\substack{h\in[0,h_0],\, x\in B,\\ k\in\{1,2,3\}}} \left\{|D^{[k]}\varphi(h,x)|\right\},\\
&\qquad \max_{h\in[0,3/2]} \left\{|\varphi'_x(h,x)|\right\},\ \max_{h\in[0,h_0]} \left\{|\varphi^{[4]}_{txxx}(h,x)|\right\}\Bigg\},\\
&\mathrm{C}_{\min} \leq \min_{x\in\gamma(\mathbb{R})} \{|f(x)|_2^2\},\\
&\mathrm{C}_\tau \geq \max_{\substack{s\in[0,1],\, k\in\{1,2\},\\ c\in \overline{B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon^\star)/2}}}} \left\{|D^{[k]}\tau(s,c)|\right\},\\
&\mathrm{C}_E \geq \max\{|E'(s)|, s\in[0,1]\},\\
&\mathrm{C}_\psi \geq \max_{\substack{h\in[0,h_0],\, x\in B\\ k\in\{1,2,3\}}} \left\{|D^{[k]}\psi(h,x)|\right\}.
\end{aligned}
\right\} \tag{2.8}
$$

Here $D^{[k]}$ is the k-th Fréchet differential. Note that an upper bound of a type C_ψ could be given simply using (2.1) and constants $\mathrm{C}_\varphi, \mathrm{C}_\Upsilon$. Next, let us have $\delta > 0, \mu \in (0,1)$ and introduce

$$
d_m := d_m(p,\delta,\mu) := \frac{\mu - \frac{\mathrm{C}_\tau \delta}{m^{p-1}}}{m(m-1)},
$$

for

$$
m \geq m_0(p,\delta,\mu) := \max\left\{ \left\lceil \frac{2}{h_0} \right\rceil, \left\lceil \left(\frac{\delta}{\delta_{\mathrm{re}}(\varepsilon^\star)}\right)^{1/p} \right\rceil, \left\lfloor \left(\frac{\mathrm{C}_\tau\delta}{\mu}\right)^{\frac{1}{p-1}} \right\rfloor + 1 \right\},
$$

where $\lceil x \rceil := \min\{k\in\mathbb{Z} : k \geq x\}$ and $\lfloor x \rfloor := -\lceil -x \rceil$ for any $x\in\mathbb{R}$. Further

$$
\left.
\begin{aligned}
\mathcal{I}_m := \mathcal{I}_m(p,\delta,\mu) &:= \left(\frac{1}{m} - d_m, \frac{1}{m} + d_m\right),\\
\mathcal{B}_m := \mathcal{B}_m(p,\delta) &:= B_{N-1}^{\delta/m^p},\\
\mathcal{H}_m := \mathcal{H}_m(p,\delta,\mu) &:= \mathcal{I}_m \times \mathbb{R} \times \mathcal{B}_m,
\end{aligned}
\right\} \tag{2.9}
$$

also for $m \geq m_0$.

The simple goal of these complicated assumptions is that for $(h,s,c)\in\mathcal{H}_m$ it is straightforward to show

$$
d_m > 0, \quad \mathcal{I}_m \subset (0,h_0], \quad c \in B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon^\star)},
$$

and

$$\frac{1-\mu}{m} < \bar{\Delta}_m < \frac{1+\mu}{m}. \tag{2.10}$$

Theorem 2.3. *Choose any constants* $\mathrm{C}_X, \mathrm{C}_\Delta$ *such that*

$$\mathrm{C}_X > \overline{\mathrm{C}}_X := \mathrm{C}_\varphi \mathrm{C}_\Upsilon, \qquad \mathrm{C}_\Delta > \overline{\mathrm{C}}_\Delta := \frac{N\mathrm{C}_\varphi^3 \mathrm{C}_\Upsilon}{\mathrm{C}_{\min}}, \tag{2.11}$$

Fix $\delta > 0$, *then for every* m *large,* μ *small enough and* $(h, s, c) \in \mathcal{H}_m(p, \delta, \mu)$ *there exists a unique pair* $(X_m, \Delta_m) = (X_m(h, s, c), \Delta_m(h, s, c))$ *such that*

$$F(X_m, \Delta_m) = F_m(h, s, c, X_m(h, s, c), \Delta_m(h, s, c)) = 0$$

and

$$|X_m - \bar{X}_m| < \mathrm{C}_X/m^p, \quad |\Delta_m - \bar{\Delta}_m| < \mathrm{C}_\Delta/m^p. \tag{2.12}$$

Moreover the functions X_m, Δ_m *are* C^3*–smooth in its arguments and*

$$(X_m, \Delta_m)(h, s+1, c) = (X_m, \Delta_m)(h, s, c), \quad (h, s, c) \in \mathcal{H}_m. \tag{2.13}$$

Proof. The proof is divided into several steps. Two main parts are the following ones:

Part 1. The solution X_m close to $\bar{X}_m$ of $G_m(h, s, c, X) = 0$ is found.

Part 2. We solve $H_m(h, s, c, X_m(h, s, c), \Delta) = 0$ for Δ near $\bar{\Delta}_m$.

These parts are handled using Lemma 2.1 and contain four steps.

Step 1.1. We show that

$$|G_m(h, s, c, \bar{X}_m)| \leq \mathrm{C}_\Upsilon h^{p+1} \tag{2.14}$$

is valid for all $(h, s, c) \in \mathcal{H}_m$ and m large enough. From (2.1) we have for $j = 1, \cdots, m-1$ if m is large enough that

$$\begin{aligned}|(G_m(h, s, c, \bar{X}_m))^j| &= |(\psi(h, \bar{x}^{j-1}) - \varphi(h, \bar{x}^{j-1}))| \\ &\leq h^{p+1}|\Upsilon(h, \bar{x}^{j-1})| \leq \mathrm{C}_\Upsilon h^{p+1}\end{aligned}$$

where $\bar{x}^0 := \xi(s,c)$. Indeed, noting that

$$\delta/m^p \le \min\{R/\mathrm{C}_\varphi, \delta_{\mathrm{re}}(\varepsilon^\star)/2\}, \text{ and } jh \le (m-1)\left(\frac{1}{m} + d_m\right) \le \frac{3}{2}$$

are valid for m large enough using (2.8) we get that

$$\begin{aligned} |\bar{x}^j - \gamma(jh+s)| &= \left| \int_0^1 \varphi_x'(jh, \gamma(s) + \vartheta E(s)c) E(s)c \, \mathrm{d}\vartheta \right| \\ &\le \mathrm{C}_\varphi |E(s)c| \le \mathrm{C}_\varphi |E(s)c|_2 = \mathrm{C}_\varphi |c|_2 \le \mathrm{C}_\varphi \delta/m^p \le R. \end{aligned}$$

Hence using (2.3) we have

$$\bar{x}^j = \varphi(jh, \xi) \in B \text{ for } j = 0, 1, \cdots, m-1, \tag{2.15}$$

and so $|\Upsilon(h, \bar{x}^{j-1})| \le \mathrm{C}_\Upsilon$ and we are done.

Step 1.2. We show that for any $\mu_1 \in (0,1)$

$$\left| D_X G_m(h,s,c,\bar{X}_m)^{-1} \right| \le \frac{\mathrm{C}_\varphi m}{1-\mu_1} \tag{2.16}$$

holds if $(h,s,c) \in \mathcal{H}_m$, and m is large enough (the main point is of course that the lower threshold of m-s depends also on μ_1, its limit is ∞ as $\mu_1 \to 0^+$ – from now on we omit remarks of this type).

Using (2.1) again we get $D_X G_m(h,s,c,\bar{X}_m)[Y] = AY + BY$ where

$$\begin{aligned} AY &:= \big(-y^1, \varphi_x'(h,\bar{x}^1)y^1 - y^2, \varphi_x'(h,\bar{x}^2)y^2 - y^3, \cdots \\ &\qquad \cdots, \varphi_x'(h,\bar{x}^{m-2})y^{m-2} - y^{m-1} \big), \\ BY &:= \left(0, h^{p+1}\Upsilon_x'(h,\bar{x}^1)y^1, h^{p+1}\Upsilon_x'(h,\bar{x}^2)y^2, \cdots, h^{p+1}\Upsilon_x'(h,\bar{x}^{m-2})y^{m-2}\right). \end{aligned}$$

Now $AY = Z$ is solvable. Straightforward computation shows

$$\left.\begin{aligned} y^1 &= -z^1 \\ y^j &= -z^j - \sum_{r=1}^{j-1} \varphi_x'(rh, \bar{x}^{j-r}) z^{j-r}, \quad j = 2, \cdots, m-1. \end{aligned}\right\} \tag{2.17}$$

Therefore $|A^{-1}Z| \le \mathrm{C}_\varphi m$ (because (2.17) implies $|y^j| \le (1 + (m-2)\mathrm{C}_\varphi)|Z|$ for $j =$

$1, \cdots, m-1$, noticing $\mathrm{C}_\varphi \geq 1$ and (2.15) we arrive at the statement). Next we obtain in a moment also $|BY| \leq \mathrm{C}_\Upsilon h^{p+1}$ ((2.15) is used again). Now using

$$h < \frac{1}{m} + d_m < \frac{1+\mu}{m} \tag{2.18}$$

we get

$$|A^{-1}B| \leq \mathrm{C}_\varphi m \mathrm{C}_\Upsilon h^{p+1} < \frac{\mathrm{C}_\varphi \mathrm{C}_\Upsilon (1+\mu)^{p+1}}{m^p}$$

and so we have $|A^{-1}B| \leq \mu_1 < 1$ if m is large enough. Lemma 1.22 implies the invertibility of $A+B$ and also that

$$\left|(A+B)^{-1}\right| \leq \frac{|A^{-1}|}{1-|A^{-1}B|} \leq \frac{\mathrm{C}_\varphi m}{1-\mu_1}$$

and we have arrived at (2.16).

Step 1.3. We show that for any $\mu_2 > 0$ we have

$$|D_X G_m(h,s,c,X_1) - D_X G_m(h,s,c,X_2)| \leq \frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m} |X_1 - X_2| \tag{2.19}$$

for all $X_1, X_2 \in \overline{B(\bar{X}_m, R/2)}$, $(h,s,c) \in \mathcal{H}_m$ and m large enough.

At first notice that from

$$\begin{aligned} \varphi(h,x) &= \varphi(0,x) + \int_0^1 \frac{\partial}{\partial \eta}(\varphi(\eta h, x))\mathrm{d}\eta \\ &= x + h\int_0^1 \varphi'_t(\eta h, x)\mathrm{d}\eta \end{aligned}$$

we have

$$\varphi''_{xx}(h,x) = h\int_0^1 \varphi'''_{txx}(\eta h, x)\mathrm{d}\eta$$

which readily implies (cf. (2.8))

$$|\varphi'_x(h,x_1) - \varphi'_x(h,x_2)| \leq h\mathrm{C}_\varphi |x_1 - x_2| \tag{2.20}$$

for all x_1, x_2 such that $x_1 + \vartheta(x_2 - x_1) \in B, \vartheta \in [0,1]$.

For m large enough we have that

$$\forall X_1, X_2 \in \overline{B(\bar{X}_m, R/2)} \,:\, x_1^j + \vartheta(x_2^j - x_1^j) \in B, \quad j = 1, \cdots, m-1. \tag{2.21}$$

This follows from the following considerations. The condition

$$\delta/m^p \le \min\{R/2\mathrm{C}_\varphi, \delta_{\mathrm{re}}(\varepsilon^\star)/2\}$$

is fulfilled for m large enough, this implies that $|\bar{x}^j - \gamma(jh+s)| < R/2$ (similar considerations as we obtained (2.15)). Now

$$\begin{aligned} &|x_1^j + \vartheta(x_2^j - x_1^j) - \gamma(jh+s)| \\ &\le (1-\vartheta)|x_1^j - \bar{x}^j| + \vartheta|x_2^j - \bar{x}^j| + |\bar{x}^j - \gamma(jh+s)| \\ &< (1-\vartheta)\frac{R}{2} + \vartheta\frac{R}{2} + \frac{R}{2} = R \end{aligned}$$

so from (2.3) we have $x_1^j + \vartheta(x_2^j - x_1^j) \in B$ which is exactly (2.21).

For such an X_1, X_2 using (2.1) we derive that

$$\begin{aligned} &\left(D_X G_m(h,s,c,X_1) - D_X G_m(h,s,c,X_2)\right)[Y] = \\ &\Big(0, \left(\varphi_x'(h,x_1^1) - \varphi_x'(h,x_2^1)\right) y^1, \left(\varphi_x'(h,x_1^2) - \varphi_x'(h,x_2^2)\right) y^2, \cdots \\ &\cdots, \left(\varphi_x'(h,x_1^{m-2}) - \varphi_x'(h,x_2^{m-2})\right) y^{m-2}\Big) \\ &+\Big(0, h^{p+1}\left(\Upsilon_x'(h,x_1^1) - \Upsilon_x'(h,x_2^1)\right) y^1, h^{p+1}\left(\Upsilon_x'(h,x_1^2) - \Upsilon_x'(h,x_2^2)\right) y^2, \cdots \\ &\cdots, h^{p+1}\left(\Upsilon_x'(h,x_1^{m-2}) - \Upsilon_x'(h,x_2^{m-2})\right) y^{m-2}\Big). \end{aligned}$$

Using (2.20) and (2.8) we obtain

$$|D_X G_m(h,s,c,X_1) - D_X G_m(h,s,c,X_2)| \le h(\mathrm{C}_\varphi + h^p \mathrm{C}_\Upsilon)|X_1 - X_2|.$$

Note again that (2.18) is valid, therefore for every m large enough we have

$$h(\mathrm{C}_\varphi + h^p\mathrm{C}_\Upsilon) < \frac{(1+\mu)\mathrm{C}_\varphi + \frac{(1+\mu)^{p+1}\mathrm{C}_\Upsilon}{m^p}}{m} \le \frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m}$$

and we have obtained exactly (2.19).

Step 1.4. Now the final step of the first part is coming. To fit into the framework of Lemma 2.1 with an equation $G_m(h,s,c,X)=0$ set

$$\left.\begin{aligned} &U:=\mathcal{H}_m, V:=\mathbb{R}^{N(m-1)}, x=(h,s,c), \bar{y}(x):=\bar{X}_m(h,s,c),\\ &\alpha:=\frac{\mathrm{C}_\Upsilon}{m^{p+1}},\ \beta:=\frac{\mathrm{C}_\varphi m}{1-\mu_1},\ l:=\frac{(1+\mu)\mathrm{C}_\varphi+\mu_2}{m},\ \varrho:=\frac{\mathrm{C}_X}{m^p}. \end{aligned}\right\} \tag{2.22}$$

It has to be noted that for m large $\mathrm{C}_X/m^p \le R$ is valid and so (2.4) holds on $\overline{B(\bar{y}(x),\varrho)}$. Conditions (2.5) and (2.6) has to be fulfilled. For (2.5) pick $\mu_3\in(0,1)$, then for m large enough we get

$$\beta l\varrho = \frac{(1+\mu)\mathrm{C}_\varphi^2+\mu_2\mathrm{C}_X\mathrm{C}_\varphi}{m^p} \le \mu_3 < 1.$$

Further using (2.18) we get

$$\frac{\alpha\beta}{\varrho(1-\beta l\varrho)} < \frac{\mathrm{C}_\varphi\mathrm{C}_\Upsilon(1+\mu)^{p+1}}{\mathrm{C}_X(1-m_1)(1-\mu_3)},$$

so (2.6) in this setting will be valid if

$$\mathrm{C}_\varphi\mathrm{C}_\Upsilon\frac{(1+\mu)^{p+1}}{(1-\mu_1)(1-\mu_3)} < \mathrm{C}_X. \tag{2.23}$$

According to the assumption $\overline{\mathrm{C}}_X < \mathrm{C}_X$ and that $\frac{(1+\mu)^{p+1}}{(1-\mu_1)(1-\mu_3)} \to 1^+$ as $\mu,\mu_1,\mu_3\to 0^+$, there are always such suitably small parameters $\mu,\mu_1,\mu_3\in(0,1)$ that (2.23) is valid. Therefore Lemma 2.1 can be used (the remaining assumptions are trivially satisfied) and gives a unique element $X_m(h,s,c)\in\overline{B(\bar{X}_m,\mathrm{C}_X/m^p)}$ such that

$$G_m(h,s,c,X_m(h,s,c))=0.$$

Moreover X_m is C^3–smooth, $|X_m-\bar{X}_m|<\mathrm{C}_X/m^p$ and

$$|D_XG_m(h,s,c,X_m)^{-1}| \le \frac{\beta}{1-\beta l\varrho} \le \frac{\mathrm{C}_\varphi m}{(1-\mu_1)(1-\mu_3)}.$$

Step 2.1. Set

$$\left.\begin{aligned} z(h,s,c,\Delta) :&= H_m(h,s,c,X_m(h,s,c),\Delta)\\ &= \left\langle \psi(\Delta,x_m^{m-1})-\gamma(s), f(\gamma(s))\right\rangle. \end{aligned}\right\} \tag{2.24}$$

We show that for any $\mu_4 > 0$ we have

$$|z(h,s,c,\bar{\Delta}_m)| \leq \frac{N\mathrm{C}_\varphi^2\mathrm{C}_X + \mu_4}{m^p} \tag{2.25}$$

for all $(h,s,c) \in \mathcal{H}_m$ and m large enough. At first note that

$$z(h,s,c,\bar{\Delta}_m) = \big\langle \varphi(\bar{\Delta}_m, \bar{x}^{m-1}) - \gamma(s), f(\gamma(s))\big\rangle$$
$$+\big\langle \varphi(\bar{\Delta}_m, x_m^{m-1}) - \varphi(\bar{\Delta}_m, \bar{x}^{m-1}) + \bar{\Delta}_m^{p+1}\Upsilon(\bar{\Delta}_m, x_m^{m-1}), f(\gamma(s))\big\rangle$$

where the first term vanishes because of Lemma 2.2. From (2.10) we infer $\bar{\Delta}_m \in (0, h_0/2)$ for m large enough. Next

$$|\varphi(\bar{\Delta}_m, x_m^{m-1}) - \varphi(\bar{\Delta}_m, \bar{x}^{m-1})| \leq \mathrm{C}_\varphi |x_m^{m-1} - \bar{x}^{m-1}| < \mathrm{C}_\varphi\mathrm{C}_X/m^p,$$
$$|\bar{\Delta}_m^{p+1}\Upsilon(\bar{\Delta}_m, x_m^{m-1})| \leq \frac{(1+\mu)^{p+1}\mathrm{C}_\Upsilon}{m^{p+1}}.$$

From $|\langle a,b\rangle| \leq N|a||b|$ and $\varphi_t'(0,x) = f(x)$ we obtain

$$|z(h,s,c,\bar{\Delta}_m)| \leq \frac{N\mathrm{C}_\varphi\left(\mathrm{C}_\varphi\mathrm{C}_X + \frac{(1+\mu)^{p+1}\mathrm{C}_\Upsilon}{m}\right)}{m^p}.$$

For m large enough $\frac{N\mathrm{C}_\varphi(1+\mu)^{p+1}\mathrm{C}_\Upsilon}{m} \leq \mu_4$ is valid, therefore (2.25) holds.

Step 2.2. We show for any $\mu_5 > 0$ that

$$|D_\Delta z(h,s,c,\bar{\Delta}_m)^{-1}| \leq \frac{1+\mu_5}{\mathrm{C}_{\min}} \tag{2.26}$$

where $(h,s,c) \in \mathcal{H}_m$ and m is large enough. Straightforward computation yields

$$D_\Delta z(h,s,c,\Delta_m) = |f(\gamma(s))|_2^2 + w_m(h,s,c)$$

where

$$\begin{aligned} w_m(h,s,c) :=& \Big\langle f(\varphi(\bar{\Delta}_m, x_m^{m-1})) - f(\varphi(\bar{\Delta}_m^0, \bar{x}^{m-1,0})) \\ & + \bar{\Delta}_m^{p+1}\Upsilon_h'(\bar{\Delta}_m, x_m^{m-1}), f(\gamma(s))\Big\rangle, \\ \bar{\Delta}_m^0 :=& \bar{\Delta}_m(h,s,0) = 1-(m-1)h, \\ \bar{x}^{m-1,0} :=& \bar{x}^{m-1}(h,s,0) = \gamma(s+(m-1)h). \end{aligned}$$

Elementary considerations shows that

$$|w_m| \le \frac{N\mathrm{C}_\varphi^2\delta(\mathrm{C}_X + \delta(\mathrm{C}_\tau + \sqrt{N}\mathrm{C}_\varphi))}{m^p}$$

therefore for m large enough we obtain

$$|D_\Delta z(h,s,c,\bar{\Delta}_m)| \ge \frac{|f(\gamma(s))|_2^2}{1+\mu_5} \ge \frac{\mathrm{C}_{\min}}{1+\mu_5}.$$

This shows (2.26) and we are done.

Step 2.3. We have

$$|D_\Delta z(h,s,c,\Delta_1) - D_\Delta z(h,s,c,\Delta_2)| \le N\mathrm{C}_\varphi\mathrm{C}_\psi|\Delta_1 - \Delta_2| \tag{2.27}$$

is valid for all $(h,s,c) \in \mathcal{H}_m, \Delta_1, \Delta_2 \in [0,h_0]$ and m large. We easily derive that

$$\begin{gathered} D_\Delta z(h,s,c,\Delta_1) - D_\Delta z(h,s,c,\Delta_2) = \\ \big\langle \psi_h'(\Delta_1, x_m^{m-1}) - \psi_h'(\Delta_2, x_m^{m-1}), f(\gamma(s))\big\rangle = \\ \left\langle \int_0^1 \psi_{hh}''(\Delta_2 + \vartheta(\Delta_1 - \Delta_2), x_m^{m-1})\mathrm{d}\vartheta, f(\gamma(s))\right\rangle (\Delta_1 - \Delta_2) \end{gathered}$$

which immediately yields (2.27).

Step 2.4. Finally we solve $z(h,s,c,\Delta)$ with Lemma 2.1 (see (2.24)). Set

$$\left.\begin{aligned} U := \mathcal{H}_m,\ V := (0,h_0),\ x := (h,s,c),\ \bar{y}(x) := \bar{\Delta}_m(h,s,c), \\ \alpha := \frac{N\mathrm{C}_\varphi^2\mathrm{C}_X + \mu_4}{m^p},\ \beta := \frac{1+\mu_5}{\mathrm{C}_{\min}},\ l := N\mathrm{C}_\varphi\mathrm{C}_\psi,\ \varrho := \mathrm{C}_\Delta/m^p. \end{aligned}\right\} \tag{2.28}$$

Note (2.10) again, so $\overline{B(\bar{\Delta}_m, \varrho)} \subset V$ holds for m large enough. Now

$$\beta l \varrho = \frac{(1+\mu_5) N \mathrm{C}_\varphi \mathrm{C}_\psi \mathrm{C}_\Delta}{m^p} \leq \mu_6 < 1$$

is valid for any $\mu_6 \in (0,1)$ if m is sufficiently large which fulfills (2.5). Now

$$\frac{\alpha\beta}{\varrho(1-\beta l \varrho)} \leq \frac{(N\mathrm{C}_\varphi^2 \mathrm{C}_X + \mu_4)(1+\mu_5)}{\mathrm{C}_\Delta(1-\mu_6)},$$

therefore (2.6) holds if

$$\frac{(N\mathrm{C}_\varphi^2 \mathrm{C}_X + \mu_4)(1+\mu_5)}{\mathrm{C}_\Delta(1-\mu_6)} < 1. \tag{2.29}$$

Because of $\overline{\mathrm{C}}_\Delta < \mathrm{C}_\Delta$ and the already proven part of our theorem – that is C_X can be chosen arbitrary close to $\overline{\mathrm{C}}_X$ for m large enough – we conclude that (2.29) can be fulfilled (with sufficiently small $\mu, \mu_4, \mu_5, \mu_6 > 0$). Now Lemma 2.1 gives a unique element $\Delta_m \in \overline{B(\bar{\Delta}_m, \mathrm{C}_\Delta / m^p)}$ with $z(h,s,c,\Delta_m) = 0$. Moreover

$$|\Delta_m - \bar{\Delta}_m| < \mathrm{C}_\Delta / m^p, \quad |D_\Delta z(h,s,c,\Delta_m)^{-1}| \leq \frac{\beta}{1-\beta l \varrho} \leq \frac{1+\mu_5}{\mathrm{C}_{\min}(1-\mu_4)}$$

are valid and the proof is finished ((2.13) is a straightforward consequence of the 1–periodicity of $G_m, \bar{X}_m, H_m, z, \bar{\Delta}_m$ in the variable s, and the uniqueness parts of the steps 1.4. and 2.4.). □

Remark 2.1. In the framework of Theorem 2.3 a natural approximation of $\mathcal{P}$ is

$$\mathcal{P}_m(h,s,c) := \psi\left(\Delta_m(h,s,c), x_m^{m-1}(h,s,c)\right).$$

Now

$$|\mathcal{P}(s,c) - \mathcal{P}_m(h,s,c)| \leq \left|\varphi(\tau,\xi) - \varphi(\Delta_m, x_m^{m-1})\right| + \left|\Delta_m^{p+1} \Upsilon(\Delta_m, x_m^{m-1})\right|.$$

Notice that

$$
\begin{aligned}
&\left|\varphi(\tau,\xi)-\varphi(\Delta_m,X_m^{m-1})\right| = \left|\varphi(\bar{\Delta}_m,\bar{x}^{m-1})-\varphi(\Delta_m,x_m^{m-1})\right| \\
&\le \left|\varphi(\bar{\Delta}_m,\bar{x}^{m-1})-\varphi(\Delta_m,\bar{x}^{m-1})\right| + \left|\varphi(\Delta_m,\bar{x}^{m-1})-\varphi(\Delta_m,x_m^{m-1})\right| \\
&\le \int_0^1 |\varphi_t'(\Delta_m+\vartheta(\bar{\Delta}_m-\Delta_m),\bar{x}^{m-1})|\mathrm{d}\vartheta|\bar{\Delta}_m-\Delta_m| \\
&+ \int_0^1 |\varphi_x'(\Delta_m,x_m^{m-1}+\vartheta(\bar{x}^{m-1}-x_m^{m-1})|\mathrm{d}\vartheta|\bar{x}^{m-1}-x_m^{m-1}|
\end{aligned}
$$

therefore $|\varphi(\tau,\xi)-\varphi(\Delta_m,X_m^{m-1})| \le \mathrm{C}_\varphi(\mathrm{C}_X+\mathrm{C}_\Delta)/m^p$ (we used (2.8) and (2.21)). In addition from (2.10) and (2.12) we have

$$
|\Delta_m| \le |\bar{\Delta}_m| + |\Delta_m-\bar{\Delta}_m| \le \frac{1+\mu}{m} + \frac{\mathrm{C}_\Delta}{m^p}
$$

so

$$
\left|\Delta_m^{p+1}\Upsilon(\Delta_m,x_m^{m-1})\right| \le \frac{\left(1+\mu+\frac{\mathrm{C}_\Delta}{m^{p-1}}\right)^{p+1}\mathrm{C}_\Upsilon}{m^{p+1}}
$$

Hence for any fixed $\mu_7>0$ we have $|\Delta_m^{p+1}\Upsilon(\Delta_m,x_m^{m-1})| \le \frac{\mu_7}{m^p}$ for every m sufficiently large.

Putting all this together we arrive at

$$
|\mathcal{P}(s,c)-\mathcal{P}_m(h,s,c)| \le \kappa/m^p \tag{2.30}
$$

where $\kappa > \overline{\kappa} := \mathrm{C}_\varphi(\overline{\mathrm{C}}_X+\overline{\mathrm{C}}_\Delta)$ is an arbitrary constant, m is sufficiently large and μ,μ_7 are small enough (c.f. (2.11)).

Remark 2.2. With a minor modifications in our settings $p \ge 1$ would be possible until now (basically to tackle the additional case $p=1$ we would need: the extension ψ to be a function defined on $[-h_0,h_0]\times\mathbb{R}^N$; enlarging constants in (2.8) by replacing $[0,h_0]$ with $[-h_0,h_0]$; suitable changes in the definitions of $d_m, m_0, \mathcal{I}_m, \mathcal{B}_m$). The fundamental difference in the case $p=1$ would be that the natural requirement $0<\Delta_m<2h$ is generally *not* satisfied, even for m large. So the last step size is inappropriate. Possible correction would be to find the right number of iterations of $\psi(h,\cdot)$ to ensure that the next iteration with a step $\hat{\Delta}$ near h (at least satisfying

$0 < \hat{\Delta} < 2h$) we hit the Poincaré section. This procedure doesn't fit to our approach based on Lemma 2.1 therefore we are not going to specify the details.

2.3 Closeness of Differentials

Now we would like to get an upper bound in the spirit of (2.30) but for various differentials $|D_v[\mathcal{P}(s,c) - \mathcal{P}_m(h,s,c)]|$ for $v \in \{h,s,c\}$. At first we upgrade Lemma 2.1. Undoubtedly it is of its own interest in this abstract setting.

Lemma 2.4. *Suppose all the assumption of Lemma 2.1. Moreover let us have* $\alpha_1, \alpha_2, l_1 \geq 0$ *such that*

$$\left.\begin{aligned} \bar{y} \in C^1(U,V) \text{ and } |\bar{y}'(x)| \leq \alpha_1, \\ |\vartheta'(x)| \leq \alpha_2,\ x \in U,\ \text{for } \vartheta(x) := F(x,\bar{y}(x)), \\ |F'_x(x,y_1) - F'_x(x,y_2)| \leq l_1|y_1 - y_2| \text{ for } x \in U, y_1, y_2 \in \overline{B(\bar{y}(x),\varrho)}. \end{aligned}\right\} \tag{2.31}$$

Then we are able to extend the results of Lemma 2.1 by an estimate

$$|\mathrm{y}'(x) - \bar{y}'(x)| \leq \varrho_1, \quad x \in U, \tag{2.32}$$

where

$$\varrho_1 := \frac{\beta}{1 - \beta l \varrho} \left(l\varrho\alpha_1 + l_1\varrho + \alpha_2 \right). \tag{2.33}$$

Proof. From the equations $F(x,\mathrm{y}(x)) = 0$ and $F(x,\bar{y}(x)) = \vartheta(x)$ after differentiation we infer for $x \in U$ that

$$\begin{aligned} \mathrm{y}'(x) &= -(F'_y(x,\mathrm{y}(x)))^{-1}F'_x(x,\mathrm{y}(x)), \\ \bar{y}'(x) &= (F'_y(x,\bar{y}(x)))^{-1}(\vartheta'(x) - F'_x(x,\bar{y}(x))). \end{aligned}$$

From now we omit $(x,\mathrm{y}(x))$ and $(x,\bar{y}(x))$, the superscript $^-$ above F will indicate the substitution of $(x,\bar{y}(x))$, otherwise we substitute $(x,\mathrm{y}(x))$. We have

$$\begin{aligned} \mathrm{y}' - \bar{y}' &= (F'_y)^{-1}\left(-F'_x - F'_y\bar{y}'\right) = (F'_y)^{-1}\left((\bar{F}'_y - F'_y)\bar{y}' - \bar{F}'_y\bar{y}' - F'_x\right) \\ &= (F'_y)^{-1}\left((\bar{F}'_y - F'_y)\bar{y}' + \bar{F}'_x - F'_x - \vartheta'\right), \end{aligned}$$

from which we get exactly (2.32) (using (2.31) and the assumptions and results of Lemma 2.1) and the proof is finished. □

Now we are going applying this lemma we continue Theorem 2.3, adopting its notations get the following statement.

Theorem 2.5. *There are constants* $\mathrm{C}_{V,v}$ *for* $V \in \{X, \Delta\}$ *and* $v \in \{h, s, c\}$ *such that*

$$\left.\begin{aligned} \left|D_v[V_m - \bar{V}_m]\right| \leq & \mathrm{C}_{V,v}/m^p, \quad V \in \{X, \Delta\}, v \in \{s, c\}, \\ \left|D_h[V_m - \bar{V}_m]\right| \leq & \mathrm{C}_{V,h}/m^{p-1}, \quad V \in \{X, \Delta\}, \end{aligned}\right\} \tag{2.34}$$

where $\delta > 0$ *is an arbitrary constant,* m *is large enough,* μ *is sufficiently small and* $(h, s, c) \in \mathcal{H}_m(p, \delta, \mu)$.

Proof. To be able to apply Lemma 2.4 twice with frameworks described in (2.22) and (2.28) we have to find additionally constants (for the sake of (2.31))

$$\alpha_1 = \alpha_1[V, v], \quad \alpha_2 = \alpha_2[V, v], \quad l_1 = l_1[V, v]$$

for all $V \in \{X, \Delta\}, v \in \{h, s, c\}$. This will be a bit sweating task.

Part 1.1 – about $\alpha_1[X, v]$ *for* $v \in \{h, s, c\}$. After differentiation we get

$$D_h(\bar{x}^j) = f(\bar{x}^j)j, \quad D_s(\bar{x}^j) = \varphi'_x(jh, \xi)(f(\gamma(s)) + E'(s)c),$$
$$D_c(\bar{x}^j) = \varphi'_x(jh, \xi)E(s)$$

for $j = 1, 2, \cdots, m-1$. Therefore (using (2.8) and that $|E(s)| \leq \sqrt{N}$)

$$|D_h(\bar{X}_m)| \leq \mathrm{C}_\varphi m, \quad |D_s(\bar{X}_m)| \leq \mathrm{C}_\varphi^2 + \mu_9, \quad |D_c(\bar{X}_m)| \leq \mathrm{C}_\varphi \sqrt{N}$$

where $\mu_9 > 0$ is an arbitrary parameter and m is large enough ($\mathrm{C}_\varphi \mathrm{C}_E \delta/m^p \leq \mu_9$ is valid for m large enough). So

$$\alpha_1[X, h] := \mathrm{C}_\varphi m, \quad \alpha_1[X, s] := \mathrm{C}_\varphi^2 + \mu_9, \quad \alpha_1[X, c] := \mathrm{C}_\varphi \sqrt{N}. \tag{2.35}$$

Part 1.2 – about $\alpha_2[X,v]$ for $v \in \{h,s,c\}$. Note that

$$\begin{aligned}\bar{G}_m^j :=& G_m(h,s,c,\bar{X}_m(h,s,c))^j = \psi(h,\bar{x}^{j-1}) - \varphi(h,\bar{x}^{j-1})\\ =& h^{p+1}\Upsilon(h,\bar{x}^{j-1}), \qquad j = 1,2,\cdots,m-1.\end{aligned}$$

This implies

$$\begin{aligned}D_h(\bar{G}_m^j) =& h^p[(p+1)\Upsilon(h,\bar{x}^{j-1})\\ &+ h(\Upsilon'_h(h,\bar{x}^{j-1}) + \Upsilon'_x(h,\bar{x}^{j-1})D_h(\bar{x}^{j-1}))],\\ D_s(\bar{G}_m^j) =& h^{p+1}\Upsilon'_x(h,\bar{x}^{j-1})D_s(\bar{x}^{j-1}),\\ D_c(\bar{G}_m^j) =& h^{p+1}\Upsilon'_x(h,\bar{x}^{j-1})D_c(\bar{x}^{j-1}).\end{aligned}$$

Using Part 1.1. of this proof and $h < \frac{1+\mu}{m}$ for $\bar{G}_m := (\bar{G}_m^1, \bar{G}_m^2, \cdots, \bar{G}_m^{m-1})$ we infer

$$|D_h(\bar{G}_m)| \le \frac{\mathrm{C}_\Upsilon(\mathrm{C}_\varphi + p + 1) + \mu_{10}}{m^p},$$
$$|D_s(\bar{G}_m)| \le \frac{\mathrm{C}_\Upsilon \mathrm{C}_\varphi^2 + \mu_{10}}{m^{p+1}}, \qquad |D_c(\bar{G}_m)| \le \frac{\mathrm{C}_\Upsilon \mathrm{C}_\varphi \sqrt{N} + \mu_{10}}{m^{p+1}}.$$

for any fixed $\mu_{10} > 0$, every m large enough and μ sufficiently small. This yields

$$\left.\begin{aligned}\alpha_2[X,h] := \frac{\mathrm{C}_\Upsilon(\mathrm{C}_\varphi + p + 1) + \mu_{10}}{m^p}, \qquad \alpha_2[X,s] := \frac{\mathrm{C}_\Upsilon \mathrm{C}_\varphi^2 + \mu_{10}}{m^{p+1}},\\ \alpha_2[X,c] := \frac{\mathrm{C}_\Upsilon \mathrm{C}_\varphi \sqrt{N} + \mu_{10}}{m^{p+1}}.\end{aligned}\right\} \tag{2.36}$$

Part 1.3 – about $l_1[X,v]$ for $v \in \{h,s,c\}$. We have in a moment that $l_1[X,v] = 0$ for $v \in \{s,c\}$. Further note at first that

$$D_h G_m(h,s,c,X_i) = (\psi_h(h,\xi), \psi_h(h,x_i^1), \cdots, \psi_h(h,x_i^{m-1}))$$

for $X_i \in \overline{B(\bar{X}_m, \mathrm{C}_X/m^p)}$, $i \in \{1,2\}$. Now for x_1, x_2 such that $x_1 + \vartheta(x_2 - x_1) \in B$ for all $\vartheta \in [0,1]$ we have

$$|\psi_h(h,x_1) - \psi_h(h,x_2)| \leq \int_0^1 |\psi''_{hx}(h, x_2 + \vartheta(x_1 - x_2))| \mathrm{d}\vartheta |x_1 - x_2| \\ \leq \mathrm{C}_\psi |x_1 - x_2|$$

which implies that $l_1[X,h] := \mathrm{C}_\psi$ is a good choice. Therefore

$$l_1[X,h] := \mathrm{C}_\psi, \quad l_1[X,s] := 0, \quad l_1[X,c] := 0. \tag{2.37}$$

Part 1.4 – determining $\mathrm{C}_{X,v}$ *for* $v \in \{h,s,c\}$. Now we are ready to apply Lemma 2.4 in a setting (2.22) extended with (2.35),(2.36) and (2.37). From (2.32) we obtain exactly (2.34) in a case $V = X, v \in \{h,s,c\}$ with

$$\begin{aligned}
\mathrm{C}_{X,h} >& \overline{\mathrm{C}}_{X,h} := \mathrm{C}_\varphi[\mathrm{C}_\varphi^2 \overline{\mathrm{C}}_X + \mathrm{C}_\psi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon(\mathrm{C}_\varphi + p + 1)], \\
\mathrm{C}_{X,s} >& \overline{\mathrm{C}}_{X,s} := \mathrm{C}_\varphi^3[\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon)], \\
\mathrm{C}_{X,c} >& \overline{\mathrm{C}}_{X,c} := \sqrt{N}\mathrm{C}_\varphi^2[\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon)]
\end{aligned}$$

for every m large enough. Indeed, for example in the case $v = h$ (others are treated similarly) we get from (2.33) for $\mu_{11} > 0$ that

$$\begin{aligned}
\left|D_h(X_m - \bar{X}_m)\right| \leq& \frac{\beta}{1 - \beta l \varrho}\Big[l\varrho\alpha_1[X,h] + l_1[X,h]\varrho + \alpha_2[X,h]\Big] \\
=& \frac{\mathrm{C}_\varphi m}{(1-\mu_1)(1-\mu_3)}\left[\frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m}\frac{\mathrm{C}_X}{m^p}\mathrm{C}_\varphi m + \mathrm{C}_\psi \frac{\mathrm{C}_X}{m^p}\right. \\
&\left. + \frac{\mathrm{C}_\Upsilon(\mathrm{C}_\varphi + p + 1) + \mu_{10}}{m^p}\right] \leq \frac{\overline{\mathrm{C}}_{X,h} + \mu_{11}}{m^{p-1}}
\end{aligned}$$

for m large and μ small enough (we have also used (2.11) from Theorem 2.3).

Part 2.1 – about $\alpha_1[\Delta, v]$ *for* $v \in \{h,s,c\}$. We easily get

$$D_h(\bar{\Delta}_m) = -m + 1, \quad D_s(\bar{\Delta}_m) = \tau'_s, \quad D_c(\bar{\Delta}_m) = \tau'_c.$$

Therefore

$$\alpha_1[\Delta,h] := m, \quad \alpha_1[\Delta,s] := \mathrm{C}_\tau, \quad \alpha_1[X,c] := \mathrm{C}_\tau. \tag{2.38}$$

Part 2.2 – about $\alpha_2[\Delta, v]$ for $v \in \{h, s, c\}$. Lemma 2.2 implies (see also the definition (2.24))

$$\begin{aligned} z(h,s,c,\bar{\Delta}_m) &= \left\langle \varphi(\bar{\Delta}_m, \bar{x}^{m-1}) - \gamma(s), f(\gamma(s)) \right\rangle + \langle w_m(h,s,c), f(\gamma(s)) \rangle \\ &= \langle w_m(h,s,c), f(\gamma(s)) \rangle \end{aligned}$$

where

$$w_m := \varphi(\bar{\Delta}_m, x_m^{m-1}) - \varphi(\bar{\Delta}_m, \bar{x}^{m-1}) + \bar{\Delta}_m^{p+1} \Upsilon(\bar{\Delta}_m, x_m^{m-1}).$$

Now

$$\begin{aligned} D_v z(h,s,c,\Delta_m) =& \langle D_v w_m, f(\gamma(s)) \rangle, \quad s \in \{h, c\}, \\ D_s z(h,s,c,\Delta_m) =& \langle D_s w_m, f(\gamma(s)) \rangle + \langle w_m, f'_x(\gamma(s)) f(\gamma(s)) \rangle. \end{aligned}$$

So at first we handle terms $D_v w_m$ for $v \in \{h, s, c\}$. Straightforward computation shows that

$$\left. \begin{aligned} D_v w_m = (A_1 + A_2) D_v \bar{\Delta}_m + (A_3 + A_4) D_v \bar{x}^{m-1} \\ + (A_5 + A_4) D_v (x_m^{m-1} - \bar{x}^{m-1}), \quad v \in \{h, s, c\} \end{aligned} \right\} \tag{2.39}$$

where

$$\begin{aligned} A_1 :=& \varphi'_t(\bar{\Delta}_m, x_m^{m-1}) - \varphi'_t(\bar{\Delta}_m, \bar{x}^{m-1}), \\ A_2 :=& (p+1) \bar{\Delta}_m^p \Upsilon(\bar{\Delta}_m, x_m^{m-1}) + \bar{\Delta}_m^{p+1} \Upsilon'_h(\bar{\Delta}_m, x_m^{m-1}), \\ A_3 :=& \varphi'_x(\bar{\Delta}_m, x_m^{m-1}) - \varphi'_x(\bar{\Delta}_m, \bar{x}^{m-1}), \\ A_4 :=& \bar{\Delta}_m^{p+1} \Upsilon'_x(\bar{\Delta}_m, x_m^{m-1}), \\ A_5 :=& \varphi'_x(\bar{\Delta}_m, x_m^{m-1}). \end{aligned}$$

Let us have $\mu_{12} > 0$, then computations as in the previous parts show that for m large and μ small enough we have

$$\begin{gathered} |A_1 + A_2| \le \frac{\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon (p+1) + \mu_{12}}{m^p}, \\ |A_3 + A_4| \le \frac{\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mu_{12}}{m^p}, \qquad |A_5 + A_4| \le \mathrm{C}_\varphi + \mu_{12} \end{gathered}$$

For the remaining parts of the right side of (2.39) we have upper bounds in (2.35), (2.38) and in the already proved case of (2.34) (c.f. Part 1.4). Putting this together we get for any $\mu_{13} > 0$ that

$$|D_h w_m| \le \frac{C_1 + \mu_{13}}{m^{p-1}}, \quad |D_s w_m| \le \frac{C_2 + \mu_{13}}{m^p}, \quad |D_c w_m| \le \frac{C_3 + \mu_{13}}{m^p},$$

where m sufficiently large, μ is small enough and

$$\begin{aligned} C_1 :=& \mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon (p+1) + \mathrm{C}_\varphi^2 \overline{\mathrm{C}}_X + \overline{\mathrm{C}}_{X,h} \mathrm{C}_\varphi, \\ C_2 :=& \left(\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon (p+1) \right) \mathrm{C}_\tau + \mathrm{C}_\varphi^3 \overline{\mathrm{C}}_X + \overline{\mathrm{C}}_{X,s} \mathrm{C}_\varphi, \\ C_3 :=& \left(\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon (p+1) \right) \mathrm{C}_\tau + \sqrt{N} \mathrm{C}_\varphi^2 \overline{\mathrm{C}}_X + \overline{\mathrm{C}}_{X,c} \mathrm{C}_\varphi. \end{aligned}$$

Furthermore for $C_4 := \mathrm{C}_\varphi \overline{\mathrm{C}}_X$ similar computations show also $|w_m| \le (C_4 + \mu_{13})/m^p$.. Therefore we can finish this step with the following choices

$$\left. \begin{array}{c} \alpha_2[\Delta, h] := \dfrac{N\mathrm{C}_\varphi C_1 + \mu_{14}}{m^{p-1}}, \qquad \alpha_2[\Delta, s] := \dfrac{N\mathrm{C}_\varphi (C_2 + C_4) + \mu_{14}}{m^p}, \\ \alpha_2[\Delta, c] := \dfrac{N\mathrm{C}_\varphi C_3 + \mu_{14}}{m^p}, \end{array} \right\} \tag{2.40}$$

where $\mu_{14} > 0$ is an arbitrary parameter, m is large and μ is small enough.

Part 2.3 – about $l_1[\Delta, v]$ for $v \in \{h, s, c\}$. For $\Delta \in \overline{B(\bar{\Delta}_m, \mathrm{C}_\Delta/m^p)}$ differentiating yields

$$\begin{aligned} D_v z(h, s, c, \Delta) =& \langle \psi_x'(\Delta, x_m^{m-1}) D_v x_m^{m-1}, f(\gamma(s)) \rangle, \quad v \in \{h, c\}, \\ D_s z(h, s, c, \Delta) =& \langle \psi_x'(\Delta, x_m^{m-1}) D_s x_m^{m-1}, f(\gamma(s)) \rangle \\ & + \langle \psi(\Delta, x_m^{m-1}) - \gamma(s), \varphi_{tx}''(s, \xi_0) \rangle. \end{aligned}$$

Note that from a triangle inequality we have

$$|D_h x_m^{m-1}| \le |D_h \bar{x}^{m-1}| + |D_h (x_m^{m-1} - \bar{x}^{m-1})| \le \mathrm{C}_\varphi m + \mathrm{C}_{X,h}/m^p,$$
$$|D_s x_m^{m-1}| \le \mathrm{C}_\varphi^2 + \mu_9 + \mathrm{C}_{X,s}/m^p, \qquad |D_c x_m^{m-1}| \le \sqrt{N} \mathrm{C}_\varphi + \mathrm{C}_{X,c}/m^p.$$

Employing Newton-Leibnitz formula straightforward computations for any $\mu_{15} > 0$

and m large enough imply that following choices are suitable

$$\left.\begin{aligned} l_1[\Delta,h] := N\mathrm{C}_\psi \mathrm{C}_\varphi^2 m + \mu_{15}, \quad l_1[\Delta,s] := N\mathrm{C}_\psi \mathrm{C}_\varphi(1+\mathrm{C}_\varphi^2) + \mu_{15}, \\ l_1[\Delta,c] := N^{3/2}\mathrm{C}_\psi \mathrm{C}_\varphi^2 + \mu_{15}. \end{aligned}\right\} \tag{2.41}$$

Part 2.4 – determining $\mathrm{C}_{\Delta,v}$ *for* $v \in \{h,s,c\}$. As in the Part 1.4 we apply Lemma 2.4 in a setting (2.28) extended with (2.38),(2.40) and (2.41). From (2.32) we obtain (2.34) in a case $V = \Delta, v \in \{h,s,c\}$ with

$$\begin{aligned} \mathrm{C}_{\Delta,h} > & \overline{\mathrm{C}}_{\Delta,h} := \frac{N\mathrm{C}_\varphi[\mathrm{C}_\psi \overline{\mathrm{C}}_\Delta(1+\mathrm{C}_\varphi) + C_1]}{\mathrm{C}_{\min}}, \\ \mathrm{C}_{\Delta,s} > & \overline{\mathrm{C}}_{\Delta,s} := \frac{N\mathrm{C}_\varphi[\mathrm{C}_\psi \overline{\mathrm{C}}_\Delta(\mathrm{C}_\tau + \mathrm{C}_\varphi^2) + C_2 + C_4]}{\mathrm{C}_{\min}}, \\ \mathrm{C}_{\Delta,c} > & \overline{\mathrm{C}}_{\Delta,c} := \frac{N\mathrm{C}_\varphi[\mathrm{C}_\psi \overline{\mathrm{C}}_\Delta(\mathrm{C}_\tau + \sqrt{N}\mathrm{C}_\varphi) + C_3]}{\mathrm{C}_{\min}} \end{aligned}$$

for every m large, μ small. The proof is complete. □

Remark 2.3. Now as in the proof of Theorem 2.5 (see (2.39)) we get

$$\begin{aligned} D_v\mathcal{P}(s,c) - D_v\mathcal{P}(h,s,c) = (\bar{A}_1 - \bar{A}_2)D_v\bar{\Delta}_m + (\bar{A}_3 - \bar{A}_4)D_v\bar{x}^{m-1} \\ -(\bar{A}_5 + \bar{A}_2)D_v(\Delta_m - \bar{\Delta}_m) - (\bar{A}_6 + \bar{A}_4)D_v(X_m^{m-1} - \bar{x}^{m-1}) \end{aligned} \tag{2.42}$$

for $v \in \{h,s,c\}$, where

$$\begin{aligned} \bar{A}_1 :=& \varphi_t'(\bar{\Delta}_m, \bar{x}^{m-1}) - \varphi_t'(\Delta_m, x_m^{m-1}), \\ \bar{A}_2 :=& (p+1)\Delta_m^p \Upsilon(\Delta_m, x_m^{m-1}) + \Delta_m^{p+1}\Upsilon_h'(\Delta_m, x_m^{m-1}), \\ \bar{A}_3 :=& \varphi_x'(\bar{\Delta}_m, \bar{x}^{m-1}) - \varphi_x'(\Delta_m, x_m^{m-1}), \\ \bar{A}_4 :=& \Delta_m^{p+1}\Upsilon_x'(\Delta_m, x_m^{m-1}), \\ \bar{A}_5 :=& \varphi_t'(\Delta_m, x_m^{m-1}), \quad \bar{A}_6 := \varphi_x'(\Delta_m, x_m^{m-1}). \end{aligned}$$

From (2.10) we infer

$$|\Delta_m| \le |\bar{\Delta}_m| + |\Delta_m - \bar{\Delta}_m| \le \frac{1+\mu}{m} + \frac{\mathrm{C}_\Delta}{m^p} = \frac{1+\mu+\frac{\mathrm{C}_\Delta}{m^{p-1}}}{m}.$$

After lengthy computation for $\mu_{16} > 0$ we get

$$|\bar{A}_1 - \bar{A}_2| \leq \frac{C_5 + \mu_{16}}{m^p}, \quad |\bar{A}_3 - \bar{A}_4| \leq \frac{\mathrm{C}_\varphi \overline{\mathrm{C}}_\Delta + \mu_{16}}{m^p},$$
$$|\bar{A}_5 + \bar{A}_2| \leq \mathrm{C}_\varphi + \mu_{16}, \quad |\bar{A}_6 + \bar{A}_4| \leq \mathrm{C}_\varphi + \mu_{16},$$

where $C_5 := \mathrm{C}_\varphi(\overline{\mathrm{C}}_\Delta + \overline{\mathrm{C}}_X) + (p+1)C_5^p \mathrm{C}_\Upsilon$, m is large μ is small enough. Using in addition (2.34), (2.35) and (2.38) for remaining terms in (2.42) we finally obtain

$$\left.\begin{aligned} |D_h[\mathcal{P}(s,c) - \mathcal{P}_m(h,s,c)]| \leq& \kappa_h / m^{p-1}, \\ |D_v[\mathcal{P}(s,c) - \mathcal{P}_m(h,s,c)]| \leq& \kappa_v / m^p, \quad v \in \{s,c\}, \end{aligned}\right\} \tag{2.43}$$

where

$$\begin{aligned} \kappa_h >& C_5 + \mathrm{C}_\varphi \overline{\mathrm{C}}_\Delta \mathrm{C}_\varphi + \mathrm{C}_\varphi(\mathrm{C}_{\Delta,h} + \mathrm{C}_{X,h}), \\ \kappa_s >& C_5 \mathrm{C}_\tau + \mathrm{C}_\varphi \overline{\mathrm{C}}_\Delta \mathrm{C}_\varphi^2 + \mathrm{C}_\varphi(\mathrm{C}_{\Delta,s} + \mathrm{C}_{X,s}), \\ \kappa_c >& C_5 \mathrm{C}_\tau + \mathrm{C}_\varphi \overline{\mathrm{C}}_\Delta \mathrm{C}_\varphi \sqrt{N} + \mathrm{C}_\varphi(\mathrm{C}_{\Delta,c} + \mathrm{C}_{X,c}). \end{aligned}$$

One may wish to continue in this direction developing bounds for

$$D^2_{v_1 v_2}[\mathcal{P}_m(h,s,c) - \mathcal{P}(s,c)], \quad v_1, v_2 \in \{h,s,c\}.$$

This is quite technical (computations rather for computer), therefore we show only the key equipment namely the natural extension of Lemma 2.1 to the next level in the spirit of Lemma 2.4.

Lemma 2.6. *Suppose all the assumptions of Lemma 2.1 with $X = X_1 \times X_2 \times X_3$ ($X_i, i \in \{1,2,3\}$ are Banach spaces, and $|\cdot|_X := \max_{i \in \{1,2,3\}} |\cdot|_{X_i}$). Let us have $F \in C^r(U \times V, Z)$ for $r \geq 2$ and also $\bar{y} \in C^2(U,V)$. Suppose (like in (2.31)) that*

$$\left.\begin{aligned} |D_{x_i}\bar{y}| \leq \alpha_{1,i}, \quad |D_{x_i}\vartheta| \leq \alpha_{2,i}, \\ |F'_{x_i}(x,y_1) - F'_{x_i}(x,y_2)| \leq l_{1,i}|y_1 - y_2|, \quad x \in U, y_1, y_2 \in \overline{B(\bar{y}(x), \varrho)} \end{aligned}\right\} \tag{2.44}$$

for $i \in \{1,2,3\}$. Introduce also $\varrho_{1,i} := \frac{\beta}{1-\beta l \varrho}(l\varrho\alpha_{1,i} + l_{1,i}\varrho + \alpha_{2,i})$ accordingly to (2.32).

Further let us have

$$\left.\begin{array}{c}|D^2_{x_ix_j}\bar{y}| \le \alpha_{3,i,j}, \quad |D^2_{x_ix_j}\vartheta| \le \alpha_{4,i,j},\\ |F''_{x_iy}(x,\bar{y}(x))| \le \alpha_{5,i}, \quad |F''_{yy}(x,\bar{y}(x))| \le \alpha_6,\\ |F''_{x_ix_j}(x,y_1) - F''_{x_ix_j}(x,y_2)| \le l_{2,i,j}|y_1 - y_2|,\\ |F''_{x_iy}(x,y_1) - F''_{x_iy}(x,y_2)| \le l_{3,i}|y_1 - y_2|,\\ |F''_{yy}(x,y_1) - F''_{yy}(x,y_2)| \le l_4|y_1 - y_2|\end{array}\right\} \tag{2.45}$$

for $i, j \in \{1,2,3\}, i \le j$ *and for all* $x \in U$ *and* $y_1, y_2 \in \overline{B(\bar{y}(x),\varrho)}$. *Then*

$$|D_{x_ix_j}\mathrm{y}(x) - D_{x_ix_j}\bar{y}(x)| \le \varrho_{2,i,j}, \quad x \in U, \quad i,j \in \{1,2,3\}, i \le j, \tag{2.46}$$

where

$$\begin{aligned}\varrho_{2,i,j} :=& \frac{\beta}{1-\beta l\varrho}\Big(l\varrho\alpha_{3,i,j} + \alpha_{4,i,j} + \varrho l_{2,i,j} + \varrho_{1,j}\alpha_{5,i} + \varrho l_{3,i}(\alpha_{1,j} + \varrho_{1,j})\\ &+ \varrho_{1,i}\alpha_{5,j} + \varrho l_{3,j}(\alpha_{1,i} + \varrho_{1,i}) + \varrho_{1,i}\alpha_6\alpha_{1,j} + \varrho_{1,j}\alpha_6(\alpha_{1,i} + \varrho_{1,i})\\ &+ \varrho l_4(\alpha_{1,i} + \varrho_{1,i})(\alpha_{1,j} + \varrho_{1,j})\Big).\end{aligned}$$

Proof. Partial derivations with respect to x_i into the direction $\delta v \in X_i$ of the equations

$$F(x,\mathrm{y}(x)) = 0 \text{ and } F(x,\bar{y}(x)) = \vartheta(x)$$

gives (we use notation $\bar{F}$ from the proof of Lemma 2.4)

$$F'_{x_i}\delta v + F'_y\mathrm{y}'_{x_i}\delta v = 0, \quad \bar{F}'_{x_i}\delta v + \bar{F}'_y\bar{y}'_{x_i}\delta v = \vartheta'_{x_i}\delta v.$$

Now differentiating once more with respect to x_j into the direction $\delta w \in X_j$ we get

$$\begin{gathered}F''_{x_ix_j}[\delta v,\delta w] + F''_{x_iy}[\delta v,\mathrm{y}'_{x_j}\delta w] + F''_{yx_j}[\mathrm{y}'_{x_i}\delta v,\delta w] + F''_{yy}[\mathrm{y}'_{x_i}\delta v,\mathrm{y}'_{x_j}\delta w]\\ +F'_y\mathrm{y}''_{x_ix_j}[\delta v,\delta w] = 0,\\ \bar{F}''_{x_ix_j}[\delta v,\delta w] + \bar{F}''_{x_iy}[\delta v,\bar{y}'_{x_j}\delta w] + \bar{F}''_{yx_j}[\bar{y}'_{x_i}\delta v,\delta w] + \bar{F}''_{yy}[\bar{y}'_{x_i}\delta v,\bar{y}'_{x_j}\delta w]\\ +\bar{F}'_y\bar{y}''_{x_ix_j}[\delta v,\delta w] = \vartheta''_{x_ix_j}[\delta v,\delta w].\end{gathered}$$

Therefore as in the proof of Lemma 2.4 we infer

$$
\begin{gathered}
(\mathrm{y}''_{x_i x_j} - \bar{y}''_{x_i x_j})[\delta v, \delta w] \\
= (F'_y)^{-1}\Big\{(\bar{F}'_y - F'_y)\bar{y}''_{x_i x_j} + F'_y \mathrm{y}''_{x_i x_j} - \bar{F}'_y \bar{y}''_{x_i x_j}\Big\}[\delta v, \delta w] \\
= (F'_y)^{-1}\Big\{\big[(\bar{F}'_y - F'_y)\bar{y}''_{x_i x_j} - \vartheta''_{x_i x_j} + (\bar{F}''_{x_i x_j} - F''_{x_i x_j})\big][\delta v, \delta w] \\
+\bar{F}''_{x_i y}[\delta v, (\bar{y}'_{x_j} - \mathrm{y}'_{x_j})\delta w] + (\bar{F}''_{x_i y} - F''_{x_i y})[\delta v, \mathrm{y}'_{x_j}\delta w] \\
+\bar{F}''_{y x_j}[(\bar{y}'_{x_i} - \mathrm{y}'_{x_i})\delta v, \delta w] + (\bar{F}''_{y x_j} - F''_{y x_j})[\mathrm{y}'_{x_i}\delta v, \delta w] \\
+\bar{F}''_{yy}[(\bar{y}'_{x_i} - \mathrm{y}'_{x_i})\delta v, \bar{y}'_{x_j}\delta w] + \bar{F}''_{yy}[\mathrm{y}'_{x_i}\delta v, (\bar{y}'_{x_j} - \mathrm{y}'_{x_j})\delta w] \\
+(\bar{F}''_{yy} - F''_{yy})[\bar{y}'_{x_i}\delta v, \bar{y}'_{x_j}\delta w]\Big\}.
\end{gathered}
$$

Now using the symmetry of the second derivatives, switching to the norms and employing the assumptions of the theorem the final statement (2.46) follows and the proof is finished. □

Now we show a sketch of one possible application of Lemma 2.6. Let the equation $G_m(h, s, c, X) = 0$ be in the role of $F(x_1, x_2, x_3, y) = 0$ with a basic framework given in (2.22). We deal only with the case $i = j = 3$, when we are looking for a bound of $|D^2_{cc}X_m - D^2_{cc}\bar{X}_m|$. The proof of Theorem 2.5 – namely (2.35),(2.36) and (2.37) – using notations of (2.44) implies

$$
\alpha_1 = \sqrt{N}\mathrm{C}_\varphi, \quad \alpha_+2 = \frac{\sqrt{N}\mathrm{C}_\varphi \mathrm{C}_\Upsilon + \mu_{10}}{m^{p+1}}, \quad l_1 = 0
$$

needed in (2.44). Remaining constants in (2.45), skipping the details of the lengthy computation, are

$$
\begin{gathered}
\alpha_3 = N\mathrm{C}_\varphi, \quad \alpha_4 = \frac{N(1+\mu)^{p+1}\mathrm{C}_\varphi \mathrm{C}_\Upsilon(1+\mathrm{C}_\varphi)}{m^{p+1}}, \quad \alpha_5 = 0, \\
\alpha_6 := \frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m}, \quad l_2 = l_3 = 0, \quad l_4 = \frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m}.
\end{gathered}
$$

Now application of Lemma 2.6 yields that for $\mathrm{C}_{X,cc} > \overline{\mathrm{C}}_{X,cc}$, m large and μ small enough we have

$$
|D^2_{cc}X_m - D^2_{cc}\bar{X}_m| \le \mathrm{C}_{X,cc}/m^p \tag{2.47}
$$

where $\overline{\mathrm{C}}_{X,cc} := \mathrm{C}^2_\varphi[N\mathrm{C}_\varphi\overline{\mathrm{C}}_X + N\mathrm{C}_\Upsilon(1+\mathrm{C}_\varphi) + 2\sqrt{N}\mathrm{C}_\varphi\overline{C}_{X,c} + N\mathrm{C}^2_\varphi\overline{\mathrm{C}}_X]$.

Similarly it is possible to handle the equation $z(h,s,c,\Delta)=0$ in a setting (2.28). From (2.38),(2.40) and (2.41) we get

$$\alpha_1 = \mathrm{C}_\tau, \quad \alpha_2 = \frac{N\mathrm{C}_\varphi C_3 + \mu_{14}}{m^p}, \quad l_1 = N^{3/2}\mathrm{C}_\varphi^2\mathrm{C}_\psi + \mu_{15}.$$

Omitting again the details we get for $\mu_{17} > 0$, m large and μ small enough that

$$\alpha_3 = \mathrm{C}_\tau, \quad \alpha_4 = \frac{N(p+1)p\mathrm{C}_\varphi \mathrm{C}_\Upsilon + \mu_{17}}{m^{p-1}},$$
$$\alpha_5 = N^{3/2}\mathrm{C}_\varphi^2\mathrm{C}_\psi + \mu_{17}, \quad \alpha_6 = N\mathrm{C}_\varphi\mathrm{C}_\psi,$$
$$l_2 = N^2\mathrm{C}_\varphi^2\mathrm{C}_\psi(\mathrm{C}_\varphi + 1) + \mu_{17}. \quad l_3 = \alpha_5, \quad l_4 = \alpha_6$$

So Lemma 2.6 gives

$$|D_{cc}^2\Delta_m - D_{cc}^2\bar{\Delta}_m| \leq \mathrm{C}_{\Delta,cc}/m^{p-1} \tag{2.48}$$

for m large enough where $\mathrm{C}_{\Delta,cc} > \overline{\mathrm{C}}_{\Delta,cc} := \frac{N(p+1)p\mathrm{C}_\varphi\mathrm{C}_\Upsilon}{\mathrm{C}_{\min}}$.

Now as in the Remark 2.1 it would be possible to derive

$$|D_{cc}^2\mathcal{P}_m(h,s,c) - D_{cc}^2\mathcal{P}(s,c)| \leq C/m^{p-1}$$

for some constant C. Instead of this we show a weaker result, namely that

$$|D_{cc}^2\mathcal{P}_m(h,s,c)|$$

is uniformly bounded for every m large enough (uniformity is related to m–s).

Differentiation yields

$$D_{cc}^2\mathcal{P}_m(h,s,c)[\delta v,\delta w] = \psi_{hh}''(\Delta_m, x_m^{m-1})[D_c\Delta_m\delta v, D_c\Delta_m\delta w]$$
$$+\psi_{hx}''(\Delta_m, x_m^{m-1})[D_c\Delta_m\delta v, D_c x_m^{m-1}\delta w] + \psi_h'(\Delta_m, x_m^{m-1})D_{cc}^2\Delta_m[\delta v,\delta w]$$
$$+\psi_{xh}''(\Delta_m, x_m^{m-1})[D_c x_m^{m-1}\delta v, D_c\Delta_m\delta w]$$
$$+\psi_{xx}''(\Delta_m, x_m^{m-1})[D_c x_m^{m-1}\delta v, D_c x_m^{m-1}\delta w] + \psi_x'(\Delta_m, x_m^{m-1})D_{cc}^2 x_m^{m-1}[\delta v,\delta w].$$

Switching to the norms, using (2.8), (2.34), (2.47) and (2.48) after some computations we obtain

$$|D_{cc}^2\mathcal{P}_m(h,s,c)| \leq C_6 \tag{2.49}$$

for $C_6 > \overline{C}_6 := \mathrm{C}_\psi\big[(\mathrm{C}_\tau + \sqrt{N}\mathrm{C}_\varphi)^2 + \mathrm{C}_\tau + N\mathrm{C}_\varphi\big]$, large m and small μ.

2.4 A Closed Curve for a Discrete Dynamics

The *nondegeneracy* condition of γ

$$1 \text{ is a simple eigenvalue of } \varphi'_x(1, \xi_0) \tag{2.50}$$

is in the central role in this section.

The word *simple* means that the algebraic multiplicity of the eigenvalue 1 is one, in other words $\lambda = 1$ is a simple root of the characteristic polynomial

$$\det\big(\lambda\mathbb{I} - \varphi'_x(1, \xi_0)\big).$$

Noting

$$\varphi'_x(1, \gamma(s)) = Q\varphi'_x(1, \xi_0)Q^{-1}, \quad Q := \varphi'_x(s, \xi_0), \quad s \in \mathbb{R}$$

we have that (2.50) is equivalent to

$$1 \text{ is a simple eigenvalue of } \varphi'_x(1, \gamma(s)) \tag{2.51}$$

for any $s \in \mathbb{R}$.

Introduce $\mathbb{A}_s := E(s)^T\varphi'_x(1, \gamma(s))E(s) - \mathbb{I}_{N-1}$ where $\mathbb{I}_{N-1}$ is an $(N-1) \times (N-1)$ identity matrix. Condition (2.50) implies that $\mathbb{A}_s$ is invertible. Indeed, suppose on the contrary that $\mathbb{A}_s v = 0$ for $v \in \mathbb{R}^{N-1}, v \neq 0$. Then for $w := E(s)v \neq 0$ we infer

$$\varphi'_x(1, \gamma(s))w = \alpha f(\gamma(s)) + w, \quad \text{for some } \alpha \in \mathbb{R}.$$

Using also that $\varphi'_x(1, \gamma(s))f(\gamma(s)) = f(\gamma(s))$ we get $(\mathbb{I} - \varphi'_x(1, \gamma(s)))^2 w = 0$. Therefore the geometric multiplicity of the eigenvalue 1 is at least 2 (w and $f(\gamma(s))$ are linearly independent vectors from the generalised eigenspace; the geometric multiplicity of the eigenvalue $\mu \in \sigma(T), T \in \mathcal{B}(\mathbb{R}^n)$ is the number $\dim(\mathrm{Ker}(\lambda\mathbb{I} - T))$). This is a contradiction with (2.51) (geometric multiplicity is always less than or equal to algebraic multiplicity – for more details see [35, Chapter 6 and Appendix III]).

Theorem 2.7. *Suppose that* (2.50) *holds and we have*

$$\delta > \sqrt{N}\overline{\kappa}a, \text{ where } a := \max_{s\in[0,1]} \left|\mathbb{A}_s^{-1}\right|.$$

Then for every m large enough and μ sufficiently small there is a unique function

$$\zeta_m : \mathcal{I}_m(p,\delta,\mu)\times\mathbb{R}\to\mathcal{B}_m(p,\delta)$$

such that

$$\mathcal{P}_m(h,s,\zeta_m(h,s)) = \xi(s,\zeta_m(h,s)), \quad (h,s)\in\mathcal{I}_m\times\mathbb{R}. \tag{2.52}$$

In addition ζ_m is C^3–smooth in its arguments and $\zeta_m(h,s+1)=\zeta_m(h,s)$ for all $(h,s)\in\mathcal{I}_m\times\mathbb{R}$.

Proof. Introduce $g(h,s,c) := E(s)^T\left(\mathcal{P}_m(h,s,c)-\gamma(s)\right) - c$ for $(h,s,c)\in\mathcal{H}_m$. Then it is easy to see that (2.52) is equivalent to $g(h,s,\zeta_m(h,s))=0$. To settle this we apply again Lemma 2.1 in the framework

$$U=\mathcal{I}_m\times\mathbb{R}, \quad V=\mathbb{R}^{N-1}, \quad x=(h,s), \quad \bar{y}(x)=0\in\mathbb{R}^{N-1}.$$

From (2.30) we get

$$|g(h,s,0)| = |E(s)^T||\mathcal{P}_m(h,s,0)-\mathcal{P}(s,0)| \le \frac{\sqrt{N}\kappa}{m^p}.$$

Further using $\mathcal{P}_c'(s,0) = f(\gamma(s))\tau_c'(s,0)+\varphi_x'(1,\gamma(s))E(s)$ and (2.2) it is straightforward to verify that

$$g_c'(h,s,0)=\mathbb{A}_s+W, \quad W := E(s)^T\left[\left(\mathcal{P}_m\right)_c'(h,s,0)-\mathcal{P}_c'(s,0)\right].$$

From (2.43) we have $|W|\le\frac{\sqrt{N}\kappa_c}{m^p}$. Picking up any $\mu_{18}\in[0,1)$ for every m large enough we obtain

$$|\mathbb{A}_s^{-1}W| \le \frac{a\sqrt{N}\kappa_c}{m^p} \le \mu_{18} < 1.$$

So from Lemma 1.22 we infer that $g_c'(h,s,0)$ is invertible with

$$|g_c'(h,s,0)^{-1}| \le \frac{a}{1-\mu_{18}}.$$

Next (2.49) easily gives for $c_1, c_2 \in \mathcal{B}_m$ that

$$\begin{aligned}
&|g_c'(h,s,c_1) - g_c'(h,s,c_2)| \\
= \left|E(s)^T\right| \int_0^1 &|(\mathcal{P}_m)_{cc}''(h,s,c_2+\vartheta(c_1-c_2))| \,\mathrm{d}\vartheta |c_1-c_2| \\
&\leq \sqrt{N} C_6 |c_1-c_2|.
\end{aligned}$$

In the context of the setting of Lemma (2.1) we have derived

$$\alpha = \frac{\sqrt{N}\kappa}{m^p}, \quad \beta = \frac{a}{1-\mu_{18}}, \quad l = \sqrt{N} C_6$$

and we have also $\varrho = \frac{\delta}{m^p}$. Now for any $\mu_{19} \in (0,1)$ we get

$$\beta l \varrho = \frac{a\sqrt{N} C_6 \delta}{(1-\mu_{18}) m^p} \leq \mu_{19} < 1$$

for m large enough. So (2.5) holds. Further

$$\frac{\alpha\beta}{\varrho(1-\beta l \varrho)} = \frac{\sqrt{N}\kappa a}{(1-\mu_{18})(1-\mu_{19})\delta}$$

yields that (2.6) is valid if and only if

$$\frac{\sqrt{N}\kappa a}{(1-\mu_{18})(1-\mu_{19})} < \delta.$$

This is satisfied for m large and $\kappa - \overline{\kappa}, \mu, \mu_{18}, \mu_{19}$ small enough because of (2.30) and $\delta > \sqrt{N}\overline{\kappa} a$. Application of Lemma 2.1 gives ζ_m with the desired properties and the proof is finished. □

Remark 2.4. Introducing

$$\mathcal{N}_m := \mathcal{N}_m(p,\delta) := \left\{\xi(s,c) \in \mathbb{R}^N \,:\, s \in \mathbb{R},\, c \in \mathcal{B}_m(p,\delta)\right\}$$

according to Theorem 2.7 we can state for the appropriate values of parameters that

$$\left\{x \in \mathcal{N}_m(p,\delta) \,:\, \mathcal{P}_m\left(h, \xi^{-1}(x)\right)(x) = x\right\} = \left\{\xi(s, \zeta_m(h,s)) \,:\, s \in \mathbb{R}\right\}.$$

Thorough study of the set of m–periodic points for discretized dynamics was done in [21]. Our approach implies some results also to this direction.

Theorem 2.8. *Suppose all the assumptions of Theorem 2.7 and fix any $\eta \in (0,1)$. Then for m large enough we have for every $s \in \mathbb{R}$ a unique element $h^\star(s) \in \mathcal{I}_m$ such that*

$$\Delta_m(h^\star(s), s, \zeta_m(h^\star(s), s)) = h^\star(s).$$

Further $h^\star(s+1) = h^\star(s)$ and $h^\star \in C^3(\mathbb{R}, \mathcal{I}_m^\star)$ where

$$\mathcal{I}_m^\star := \left(\frac{1}{m} - d_m^\star, \frac{1}{m} + d_m^\star\right), \quad d_m^\star := \frac{\mathrm{C}_\Delta + \mathrm{C}_\tau \delta}{m^p(m-\eta)} < d_m.$$

Therefore

$$\left\{x \in \mathcal{N}_m(p,\delta) \,:\, x = \psi^m(h,x)\right\} = \left\{\xi(s, \zeta_m(h^\star(s), s)) \,:\, s \in \mathbb{R}\right\}.$$

Proof. It is an elementary fact that for

$$\mathrm{g}(h,s) := \Delta_m(h, s\zeta_m(h,s)) - h, \quad h \in \mathcal{I}_m, s \in \mathbb{R}$$

we have

$$\mathrm{g}(h,s) = \mathrm{g}(1/m, s) + \int_{1/m}^{h} \mathrm{g}_h'(\vartheta, s)\mathrm{d}\vartheta. \tag{2.53}$$

Now (2.8) and (2.34) yields

$$\begin{aligned}|\mathrm{g}(1/m,s)| =& |\Delta_m(1/m, s, \zeta_m(1/m,s)) - \bar{\Delta}_m(1/m, s, 0)| \\ \le& |\Delta_m(1/m, s, \zeta_m(1/m,s)) - \bar{\Delta}_m(1/m, s, \zeta_m(1/m,s))| \\ &- |\bar{\Delta}_m(1/m, s, \zeta_m(1/m,s)) - \bar{\Delta}_m(1/m, s, 0)| \\ \le& \frac{\mathrm{C}_\Delta + \mathrm{C}_\tau \delta}{m^p}.\end{aligned}$$

Further

$$\mathrm{g}_h'(\vartheta, s) = (\Delta_m)_h'(\vartheta, s, \zeta_m(\vartheta, s)) + (\Delta_m)_c'(\vartheta, s, \zeta_m(\vartheta, s))(\zeta_m)_h'(\vartheta, s) - 1$$

From Theorem 2.7 (using notation for g from its proof) we infer

$$(\zeta_m)'_h(\vartheta, s) = -g'_h(\vartheta, s, \zeta_m(\vartheta, s))\big[g'_c(\vartheta, s, \zeta_m(\vartheta, s))\big]^{-1}$$

hence (cf. Lemma 2.1 and bound (2.43))

$$|(\zeta_m)'_h(\vartheta, s)| \leq \frac{a\sqrt{N}\kappa_h}{(1-\mu_{18})(1-\mu_{19})m^{p-1}}.$$

In addition using (2.34) elementary computations show

$$\begin{gathered}(\Delta_m)'_h(\vartheta, s, \zeta_m(\vartheta, s)) = (\bar{\Delta}_m)'_h(\vartheta, s, \zeta_m(\vartheta, s)) + w_m = -m + 1 + v_m,\\ v_m := (\Delta_m)'_h(\vartheta, s, \zeta_m(\vartheta, s)) - (\bar{\Delta}_m)'_h(\vartheta, s, \zeta_m(\vartheta, s)), \quad |v_m| \leq \frac{\mathrm{C}_{\Delta,h}}{m^{p-1}},\\ |(\Delta_m)'_c(\vartheta, s, \zeta_m(\vartheta, s))| \leq \mathrm{C}_\tau + \frac{\mathrm{C}_{\Delta,c}}{m^p}.\end{gathered}$$

Combining these facts we get

$$\mathrm{g}'_h(\vartheta, s) = -m + w_m, \qquad |w_m| \leq \eta \tag{2.54}$$

for every m large enough.

The relation $d^\star_m < d_m$ holds evidently for m sufficiently large. Now (2.53) after easy computations implies that

$$\left.\begin{aligned} g(h, s) < 0, \ \text{for } h \in \left[\frac{1}{m} + d^\star_m, \frac{1}{m} + d_m\right), \\ g(h, s) > 0, \ \text{for } h \in \left(\frac{1}{m} - d_m, \frac{1}{m} - d^\star_m\right]. \end{aligned}\right\} \tag{2.55}$$

Because of $g(\cdot, s) : \mathcal{I}_m \to \mathbb{R}$ is a C^1–function with properties (2.55) and (2.54) we get a unique element $h^\star(s) \in \mathcal{I}_m$ such that $g(h^\star(s), s) = 0$ moreover $h^\star(s) \in \mathcal{I}^\star_m$. Application of the implicit function theorem on the equation $g(h, s) = 0$ in the neighbourhood of the solution $(h^\star(s'), s')$ for any $s' \in \mathbb{R}$ yields also the C^3–smoothness of $h^\star : \mathbb{R} \to \mathcal{I}^\star_m$ and the proof is completed (the periodicity of $h^\star$ is straightforward). □

Remark 2.5. Usual arguments yield that for any $A_0 \in \mathcal{B}(\mathbb{R}^{N-1})$ and $r > 0$ we have that the following minimum is attained and

$$\min_{\substack{\lambda\in\mathbb{C}\setminus B_r\\ z\in\mathbb{R}^{N-1},|z|=1}} |(\lambda\mathbb{I}-A_0)z| := c(r) > 0,$$

where $B_r := \bigcup_{\mu\in\sigma(A_0)} B(\mu,r)$ and $\sigma(A_0)\subset\mathbb{C}$ is the spectrum of A_0. Therefore for any $A\in\mathcal{B}(\mathbb{R}^{N-1})$ such that $|A-A_0|<c(r)$ we have $\sigma(A)\subset B_r$ (for more general statement see [10, Corollary 2.6, pp. 470]). Indeed, for $\lambda\in\mathbb{C}\setminus B_r$ we have

$$\lambda\mathbb{I}-A = (\lambda\mathbb{I}-A_0)(I+(\lambda\mathbb{I}-A_0)^{-1}(A_0-A))$$

and

$$\left|(\lambda\mathcal{I}_m-A_0)^{-1}(A_0-A)\right| \le \left|(\lambda\mathcal{I}_m-A_0)^{-1}\right| |A_0-A| < \frac{1}{c(r)}\cdot c(r) = 1.$$

From Lemma 1.22 we get $\lambda\in\mathbb{C}\setminus\sigma(A)$ which gives $\mathbb{C}\setminus B_r\subset\mathbb{C}\setminus\sigma(A)$ and we are done. Now set

$$A_0 := E(s)^T\mathcal{P}_c'(s,0), \qquad A := E(s)^T(\mathcal{P}_m)_c'(h,s,c)$$

for any $(h,s,c)\in\mathcal{H}_m$. Then after careful computations using primary (2.43) we get $|A-A_0|<c(r)$ for m large enough. This yields

$$\sigma(A)\subset\bigcup_{\mu\in\sigma(A_0)} B(\mu,r).$$

Hence with $c=\zeta_m(h,s)$, or $h=h^\star(s)$ and $c=\zeta_m(h^\star(s),s)$, we obtain for the above detected curves also the corresponding closeness statements about their (h,s) and s dependent spectrum.

Conclusion

Hereby we would like to summarize the main new results of the thesis. We also give some short comments on the main tools of their proofs. Finally we discuss the possible continuation of the topics treated in this thesis.

In Subsection 1.2.2 under suitable conditions we proved that

$$\dot{x} \in A(t)x + f(t,x) + g(t,x,X_1)$$

has a unique γ-q. b. solution Γ_γ for $\gamma \in (\alpha,\beta), A \in \mathcal{E}_{\alpha,\beta}(\mathbb{R})$ and corresponding to suitable selectors $h \in \mathsf{H}_\varepsilon$. We introduced the set $S^\gamma_{\tau,\varepsilon}$ of points $\Gamma_\gamma(\tau,h)$ and the generalizations of the stable/unstable set of $S^\gamma_{\tau,\varepsilon}$. These were designated by symbols $M^{r,\gamma}_{\tau,\varepsilon}, r \in \{s,u\}$. Then we gave – as we had anticipated – the graph characterizations of these sets. The results were stated in Theorem 1.18 and 1.19. The key idea for their proofs was a simple transformation described in Lemma 1.17. Only one natural difficulty arose, namely the condition (1.19). In order to replace it by a more natural one a thorough discussion was made in remarks 1.8, 1.9. Basically we introduced there new selector spaces which allowed us to transform our setting to the already handled "bounded" problems in Theorem 1.18 and 1.19.

After that we gave answers to two important questions (see Subsection 1.2.3). Firstly we showed that Γ_γ is independent of γ under suitable conditions (Theorem 1.20). Then secondly, we examined differential inclusions possessing hyperbolic exponential dichotomy with several projections. Here we established conditions which ensure that the system $M^{r,\gamma_i}_{\tau,\varepsilon}, r \in \{s,u\}, i = 1, \cdots n$ inherit the adequate hierarchy of projector ranges $P^\pm_i(X)$ (cf. Theorem 1.21). The proofs of these facts were carried out by the application of Theorems 1.18, 1.19).

Chapter 1 was finished with valuable remarks on hyperbolic exponential dichotomy on $\mathbb{R}$. In Theorems 1.23, 1.24 we gave sufficient conditions for $\lambda \in \mathbb{C}$ to lie in the resolvent of $A \in \mathcal{B}(X)$ on a finite dimensional X and also for $X = \ell_p$, $p \in [1, \infty]$. These results followed from Lemma 1.22 with a properly suited framework. Then we analyzed some non-autonomous periodic ODE's. A combination of the previous results with the well-known equivalent definitions of exponential dichotomy led to some sufficient conditions for these ODE's being exponentially dichotomous (see the Subsection 1.3.3).

The main result of Chapter 2 is Theorem 2.3. It allowed us the exact analytical definition of the numerical Poincaré map $\mathcal{P}_m$ close to the classical Poincaré map $\mathcal{P}$. The proof basically relied on Lemma 2.1. After that with improvements of Lemma 2.1 – which are undoubtedly of their own interest (Lemma 2.4 and 2.6) – we stated and proved upper bounds for differentials of $\mathcal{P} - \mathcal{P}_m$.

Having at hand the properly characterized numerical Poincaré map $\mathcal{P}_m$ we devoted Section 2.4 to the naturally arising phenomenon. Namely is the curves being invariant under $\mathcal{P}_m$ in some sense. We obtained in Theorem 2.7 the existence of an h-dependent curve $(h, s) \to \zeta_m(h, s)$, (1-periodic in $s \in \mathbb{R}$) which is invariant under $\mathcal{P}_m(h, s, \cdot)$ for every (h, s) from a given subset of $[0, 1] \times \mathbb{R}$. This was proved again using Theorem 2.1 with a crucial bounds had been derived in Section 2.3. A curve of m-periodic points of the discretized system was also established in Theorem 2.8. Although this second curve had been already detected in [21], our approach has still some novelty value because we obtained qualitatively the same results using the approach through discretized Poincaré map.

Based on this summary we may conclude that the goals of the work have been fulfilled, moreover some additional questions which arose during the investigations were also successfully answered.

Besides this, the topic is far from being closed, quite the opposite is true. Our concrete future plans are

1) Examine what consequences follow for the discretized system if at some critical parameter value the continuous system undergoes a bifurcation from the periodic orbit γ. Here the starting point will be a parametric ODE where the nondegeneracy condition (2.50) of γ is altered at the critical value. The Lyapunov–Schmidt method shall hopefully lead us to some reasonable bifurcation

equation. We would like to carry out these consideration as general as possible and find some connections between the genuine and the derived bifurcation equation.

2) Turn to some concrete co-dimension one bifurcations and exhibit as much as possible for the dynamics of the discretized system.

We have been concerned about these topics since we was acquainted by the series of work [43,44,48–50] where numerical bifurcations were profoundly presented under the assumed occurrence of the bifurcation from the equilibrium in the continuous DS.

Some other possible directions would be

3) Averaging methods for general discretization procedures near the equilibrium. How does the solution of the averaged system links to the exact and numerical solution? For more see [24] where the Euler scheme was investigated.

4) An analysis of the relation between manifolds introduced in Chapter 1 and the so-called multivalued numerical schemes (see [42]).

5) A generalization of Chapter 1 to some unbounded operators, e.g. in the abstract strongly continuous semigroup setting (c.f. the thorough material [9]).

Bibliography

[1] J. P. AUBIN & A. CELLINA: Differential Inclusions, Set-Valued Maps and Viability Theory. Springer, Berlin, 1984.

[2] J. P. AUBIN & H. FRANKOWSKA: Set-Valued Analysis. Birkhäuser, Basel, 1990.

[3] B. AULBACH & T. WANNER: *Integral Manifolds for Carathéodory Type Differential Equations in Banach Spaces*, In: B. AULBACH & F. COLONIUS, *Six Lectures on Dynamical systems*, World Scientific, Singapure, 45-119.

[4] B. AULBACH & T. WANNER: *Invariant Foliations for Carathéodory Type Differential Equations in Banach Spaces*, In: V. LAKSHMIKANTHAM & A. A. MARTYNYUK: Advances in Stability Theory at the End of the 20th Century, Taylor & Francis, London, **13** (2003), 1-14.

[5] F. BATELLI & C. LAZARRI: *Exponential dichotomies, heteroclinic orbits, and Melnikov functions*, J. Diff. Equations **86** (1990), 342-366.

[6] W–J. BEYN: *On invariant closed curves for one–step methods*, Numer. Math. **51** (1987), 103-122.

[7] A. A. BOICHUK & A. A. POKUTNII: *Bounded solutions of linear differential equations in a Banach space*, Nonlinear Oscillations **9** (2006), 1-12.

[8] M. BRAUN & J. HERSHENOV: *Periodic solutions of finite difference equations*, Quart. Appl. Math. **35** (1977), 139-147.

[9] C. CHICONE & Y. LATUSHKIN: Evolution Semigroups in Dynamical Systems and Differential Equations. Math. Survey Monograph, Vol. 70, Amer. Math. Soc., Providence, 1999.

[10] S. N. CHOW, J. K. HALE: Methods of Bifurcation Theory. Springer-Verlag, New York, 1982.

[11] D. L. COHN: Measure Theory. Birkhäuser, Boston, 1980.

[12] G. COLOMBO & M. FEČKAN & B. M. GARAY: *Multivalued perturbation of a saddle dynamics*, Diff. Equ. Dyn. Sys. **18** (2010), 29-56.

[13] G. COLOMBO & M. FEČKAN & B. M. GARAY: *On the chaotic behaviour of discontinuous systems*, J. Dynamics Differential Equations **23** (2011), pp. 495-540.

[14] W. A. COPPEL: Dichotomies in Stability Theory. Springer-Verlag, Berlin, 1978.

[15] J. L. DALECKIJ & M. G. KREIN: Stability of Differential Equations in Banach Space. Amer. Math. Soc., Providence RI, 1974.

[16] K. DEIMLING: Ordinary Differential Equations in Banach Spaces. Springer, Berlin, 1977.

[17] K. DEIMLING: Nonlinear Functional Analysis. Springer-Verlag, Berlin 1985.

[18] K. DEIMLING: Multivalued Differential Equations, Walter de Gruyter. Berlin 1992.

[19] N. DUNFORD & J. T. SCHWARTZ: Linear Operators, Part 1: General Theory. Intercience Publisher, 1958.

[20] D. E. EDMUNDS & W. D. EVANS: Spectral Theory and Differential Operators. Clarendon Press, Oxford, 1987.

[21] T. EIROLA: *Two concepts for numerical periodic solutions of ODE's*, Appl. Math Comput. **31** (1989), 121-131.

[22] T. EIROLA: *Invariant curves for one step methods*, BIT **43** (1988), 113-122.

[23] M. FEČKAN: *The relation between a flow and its discretization*, Math. Slovaca **42** (1992), 123-127.

[24] M. FEČKAN: *Discretization in the method of averaging*, Proc. Amer. Math. Soc. 113 (1991), 1105-1113.

[25] M. FEČKAN & S. KELEMEN: *Multivalued Integral Manifolds in Banach Spaces*, Comm. Math. Anal. **10** (2011), 97 - 117.

[26] M. FEČKAN & S. KELEMEN: *Discretization of Poincaré Map*, submitted.

[27] B. M. GARAY: *Estimates in discretizing normally hyperbolic compact invariant manifolds of ordinary differential equations*, Computers Math. Applic. **42** (2001), 1103-1122.

[28] B. M. GARAY: *On C^j-Closeness Between the Solution Flow and its Numerical Approximation*, J. Difference Eq. Appl. **2** (1996), 67-86.

[29] M. I. GIL: Operator Functions and Localization of Spectra. Springer, Berlin, 2003.

[30] L. GRÜNE: Asymptotic Behaviour of Dynamical and Control Systems under Perturbation and Discretization. Springer, Berlin, 2002.

[31] E. HAIRER & CH. LUBICH & G. WANNER: Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations. Springer-Verlag, Berlin, 2006.

[32] J. K. HALE: Ordinary Differential Equations, Dover Publications. New York, 2009 (first published: John Wiley & Sons, 1969).

[33] M. HÉNON: *On the numerical computation of Poincaré maps*, Physica D **5** (1982), 412-414.

[34] C. J. HIMMELBERG: *Measurable relations*, Fund. Math. **87** (1975), 53-72.

[35] M. W. HIRSCH & S. SMALE: Differential Equations, Dynamical Systems and Linear Algebra, Academic Press, San Diego, 1974.

[36] M. C. IRWIN: Smooth Dynamical Systems. Academic Press, New York, 1980.

[37] L. V. KANTOROVICH & G. P. AKILOV: Functional Analysis in Normed Spaces. Moscow, 1959, translated by D. Brown, Pergamon, New York, 1964.

[38] U. KIRCHRABER & K. PALMER: Geometry in the Neighbourhood of Invariant Manifolds of Maps and Flows and Linearization. Pitman, London, 1991.

[39] P.E. KLOEDEN & V.S. KOZYAKIN: *The inflation of attractors and their discretization: the autonomous case*, Nonlinear Anal. **40** (2000), 333-343.

[40] J. KURZWEIL: Obyčejné diferenciální rovnice. (in Czech), Praha, SNTL, 1978.

[41] P. LANCESTER & M. TISMENETSKY: The Theory of Matrices, Second Edition with Application. Academic Press, Inc., 1985.

[42] F. LEMPIO & V. VELIOV: *Discrete Approximations of Differential Inclusions*, Bayreuther Mathemat. Schr. **54** (1998), 149-232.

[43] L. LÓCZI: Discretizing Elementary Bifurcations. PHD Thesis, Budapest University of Technology, 2006.

[44] L. LÓCZI & J. PÁEZ CHÁVEZ: *Preservation of bifurcations under Runge-Kutta methods*, International Journal of Qual. Theory of Dif. Eq. and Appli. **3** (2009), 81-98.

[45] M. MEDVEĎ: Fundamentals of Dynamical Systems and Bifurcation Theory. Adam Hilger, Bristol, 1992.

[46] A. ORNELAS: *Parametrization of Carathéodory multifunctions* , Rend. Sem. Mat. Univ. Padova **83** (1990), 33-44.

[47] J.M. ORTEGA: *The Newton–Kantorovich Theorem*, The Amer. Math. Monthly **75**, (1986), 658-660.

[48] J. PÁEZ CHÁVEZ: *Discretizing Dynamical Systems with Hopf-Hopf Bifurcations*, IMA J. Numer. Anal. **32** (2012), 185-201.

[49] J. PÁEZ CHÁVEZ: *Discretizing Dynamical Systems with Generalized Hopf Bifurcations*, Numer. Math. 118, **2** (2011), 229-246.

[50] J. PÁEZ CHÁVEZ: Numerical Analysis of Dynamical Systems with Codimension two Singularities. PhD Thesis, Bielefeld University, 2009.

[51] K. J. PALMER: *Exponential dichotomies and transversal homoclinic points*, J. Diff. Equations **55** (1990), 225-256.

[52] M. PARODI: La Localisation des Valeurs Caractéristiques des Matrices et ses Applications. Gauthier-Villars, Paris, 1959.

[53] CH. PUGH & M. SHUB: *C^r stability of periodic solutions and solution schemes*, Appl. Math. Lett. **1**, (1988), 281-285.

[54] A. M. SAMOILENKO & Y. V. TEPLINSKIJ: Countable Systems of Differential Equations. Brill, Utrecht, 2003.

[55] M. SHUB: Global stability of dynamical systems. Springer, Berlin, 1987.

[56] S. SMALE: *Stable manifold for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa **17** (1963), 97-116.

[57] S. SMALE: *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747-817.

[58] G. V. SMIRNOV: Introduction to the theory of differential inclusions. Americal Mathematical Society, Graduete Studies in Mathematics, **41**, 2002.

[59] A. M. STUART: *Numerical analysis of dynamical systems*, Acta Numer. **3** (1994), 467-572.

[60] A. M. STUART & A.R. HUMPHRIES: Dynamical Systems and Numerical Analysis. Cambridge Univ. Press, Cambridge, 1998.

[61] A. E. TAYLOR: Linear Functional Analysis. John Willey & Sons, Inc., New York, 1967.

[62] G. TESCHL: Jacobi Operators and Completely Integrable. Nonlinear Lattices, Amer. Math. Soc., 2000.

[63] A. A. TOLSTONOGOV: Differential Inclusions in a Banach Space. Kluwer, Dordrecht, 2000.

[64] W. TUCKER *Computing accurate Poincaré maps*, Physica D **171** (2002), 127-137.

[65] D. TURZÍK & M. DUBCOVÁ: *Stability of steady state and traveling waves solutions in coupled map lattices*, Int. J. Bifur. Chaos **18** (2008), 219-225.

[66] R. S. VARGA: Geršgorin and His Circles. Springer, Berlin, 2011.

[67] T. YAMAMOTO: *Historical developments in convergence analysis for Newton's and Newton-like methods*, J. Computational App. Math., **124**, (2000), 1-23.