

On a Natural Notion of Entropy and Independence via Fuzzy Relations

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Abstract. We revisit the concept of fuzzy entropy on fuzzy T -equivalence relations, where T is a t-norm. Existing definitions are logarithm-based and defined on finite universes. We extend this framework to arbitrary measure spaces and to a broader class of entropy functionals by replacing the logarithm with the additive generator of a continuous Archimedean t-norm. We prove that for jointly convex t-norms, this definition guarantees subadditivity, ensuring that the fuzzy mutual information derived from it is always non-negative, which allows us to introduce a natural notion of independence for fuzzy relations admitting a clean characterization in terms of the t-norm itself. We then build on this to open the door to (conditional) independence testing without a probabilistic interpretation of the data.

Keywords: Fuzzy equivalence relations · Fuzzy entropy · Mutual information · Independence testing · Jointly convex t-norms

Summary

Independence testing is central to many statistical and machine learning tasks, including causal discovery, where it enables algorithms such as PC to identify dependencies and remove spurious edges in causal graphs [2]. Existing work combining fuzzy data with causal methods typically either applies classical probabilistic techniques to fuzzy-valued data or extends causal estimands (e.g., treatment effects) to fuzzy settings. However, a formal notion of independence for data without an assumed probability distribution remains largely unexplored.

We revisit fuzzy entropy for fuzzy T -equivalence relations [3], where each relation represents similarity between instances with respect to a feature. Entropy is defined analogously to Shannon entropy by replacing probabilities with normalized fuzzy cardinalities of equivalence classes, i.e., the total membership degree of all instances to the class of a given instance divided by the number of instances. However, this formulation is restricted to finite universes and does not ensure subadditivity, which may yield negative fuzzy mutual information and limits its interpretation as a dependence measure between fuzzy relations.

Our proposal replaces the logarithm with the additive generator t of a continuous Archimedean t-norm T and extends the definition to a finite measure space $(U, \mathcal{A}, \nu)$ with $\nu(U) < +\infty$. Setting $\varphi(x) = t(x)$ for $x \in (0, 1]$ and $\varphi(0) = 0$, we define

$$FH_T(\mathcal{R}) = \frac{1}{\nu(U)} \int_U \varphi\left(\frac{|[x]_{\mathcal{R}}|}{\nu(U)}\right) d\nu(x),$$

where $\mathcal{R} = \{R_1, \dots, R_m\}$ is a finite family of measurable fuzzy T -equivalence relations on U , with $R_k : U \times U \rightarrow [0, 1]$, and

$$|[x]_{\mathcal{R}}| = \int_U T(R_1(x, y), \dots, R_m(x, y)) d\nu(y).$$

The formulation in [3] is recovered when T is the product t-norm (for which $t(x) = -\log x$), U is finite, and ν is the counting measure. However, unlike the product t-norm, jointly convex t-norms guarantee that the proposed entropy is subadditive,

$$FH_T(\mathcal{R}_1 \cup \mathcal{R}_2) \leq FH_T(\mathcal{R}_1) + FH_T(\mathcal{R}_2),$$

which guarantees non-negativity of the fuzzy mutual information

$$FMI_T(\mathcal{R}_1; \mathcal{R}_2) = FH_T(\mathcal{R}_1) + FH_T(\mathcal{R}_2) - FH_T(\mathcal{R}_1 \cup \mathcal{R}_2).$$

This parallels the role of mutual information in classical information theory and allows us to define independence between fuzzy relations via $FMI_T(\mathcal{R}_1; \mathcal{R}_2) = 0$, which is equivalent to

$$T\left(\frac{|[x]_{\mathcal{R}_1}|}{\nu(U)}, \frac{|[x]_{\mathcal{R}_2}|}{\nu(U)}\right) = \frac{|[x]_{\mathcal{R}_1 \cup \mathcal{R}_2}|}{\nu(U)}, \quad \text{for } \nu\text{-a.e. } x \in U.$$

Building on this framework, we can construct independence tests for feature variables based on fuzzy mutual information, where fuzzy relations are induced by similarity kernels. We evaluate the proposed method by adapting established mutual-information-based independence testing procedures (e.g., [1]) to the fuzzy setting and benchmarking across multiple t-norms, kernel choices, and experimental scenarios.

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