Mathematical Institute, Slovak Academy of Sciences

Structure of associative fusion functions

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Fusion functions

Function $F\colon [0,1]^n \longrightarrow [0,1]$ ($F\colon [0,1]^n \longrightarrow [0,1]^m$, $m \ll n$)

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76.50 EURO

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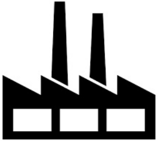

12000 cars

Fusion functions

Function $F\colon [0,1]^n \longrightarrow [0,1]$ $(F\colon [0,1]^n \longrightarrow [0,1]^m,\, m \ll n)$

1.3 average grade

Fusion functions

Representation of a group of people

Fusion functions

Representation of a group of people

8 people in the group

Fusion functions

Representation of a group of people

sum of their weights is 624.5 kg

Fusion functions

Representation of a group of people

maximum height is 187cm

Fusion functions

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Domains

- $[0,1]$, $[a,b]$
- (Bounded) lattices, posets
- Betweenness sets, etc.

Associative fusion functions

- n-ary form is uniquely given
- new inputs are added with almost no computational cost
- knowledge from the theory of semigroups
- knowledge from the theory of functional equations
- structures, results and approaches can be used for more general functions

Associative fusion functions – additional properties

- Commutativity
- Monotonicity
- Idempotency
- Continuity
- Neutral element, annihilator

Aggregation functions

Aggregation function is a non-decreasing fusion function such that

$$A(0, \ldots, 0) = 0 \text{ and } A(1, \ldots, 1) = 1$$

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Commutative, associative and monotone fusion functions (CAM fusion functions)

Triangular norm is CAM aggregation function with neutral element 1

Triangular norms – construction methods

- Isomorphism
- Additive generator (transf. of $([0, \infty], +), ([0, 1], \oplus))$
- Rotation, Rotation annihilation
- Completion methods

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- Isomorphism
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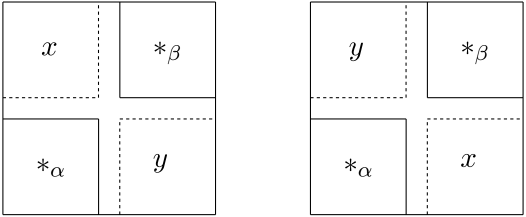
- Rotation, Rotation annihilation
- Completion methods
  - Ordinal sum
- Many others

Ordinal sum construction



$$x * y = \begin{cases} x *_\alpha y & \text{if } (x, y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha \prec \beta, \\ y & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \beta \prec \alpha. \end{cases}$$

Representation of t-norms

- The only idempotent t-norm is the minimum t-norm $T_{\mathbf{M}}$
- Each continuous t-norm can be expressed as an ordinal sum of a countable number of continuous Archimedean t-norms
- Each continuous Archimedean t-norm possesses a continuous additive generator

Triangular conorms

Triangular conorm is CAM aggregation function with neutral element 0

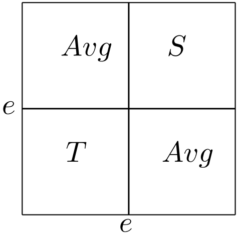
For each t-norm $T\colon [0,1]^2 \longrightarrow [0,1]$ the function $S\colon [0,1]^2 \longrightarrow [0,1]$ given by

$$S(x,y) = 1 - T(1-x, 1-y)$$

is a t-conorm

Uninorms

Uninorm is CAM aggregation function with neutral element $e \in [0, 1]$



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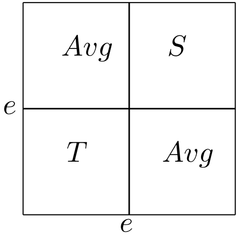
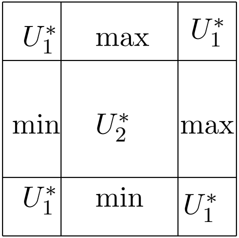
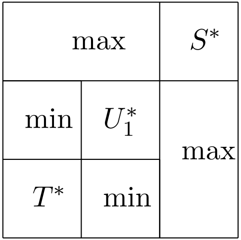


Uninorms with continuous additive generator are called representable (transformation of $([-\infty, \infty], +)$)

Uninorms with continuous underlying functions

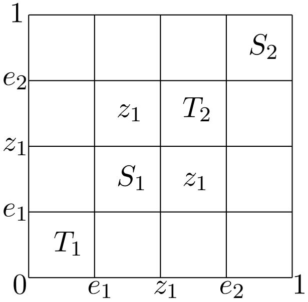




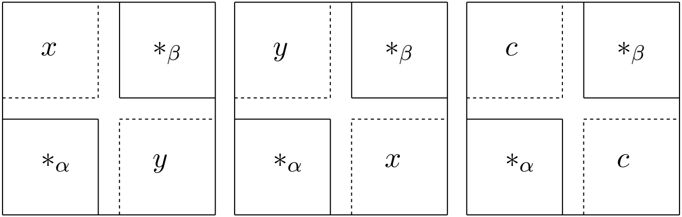
Each uninorm with **continuous** underlying functions can be expressed as an **ordinal sum** of a countable number of semigroups related to **continuous Archimedean t-norms and t-conorms, representable uninorms** and a possibly uncountable number of **trivial semigroups**.

n-Uninorms

n-uninorm is CAM aggregation function with n local neutral elements $e_i \in [z_{i-1}, z_i]$ for $i \in \{1, \ldots, n\}$



z-ordinal sum



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n-uninorms with continuous underlying functions

Each n-uninorm with **continuous** underlying functions can be expressed as a z**-ordinal sum** of a countable number of semigroups related to **continuous Archimedean t-norms and t-conorms, representable uninorms** and a possibly uncountable number of **trivial semigroups.**

n-uninorms with continuous underlying functions

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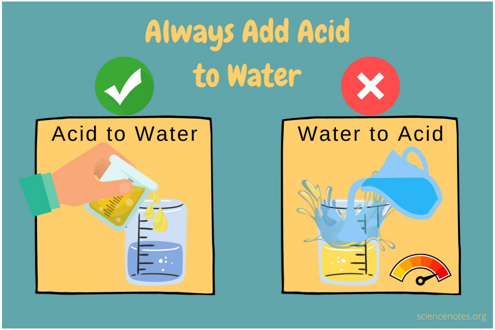
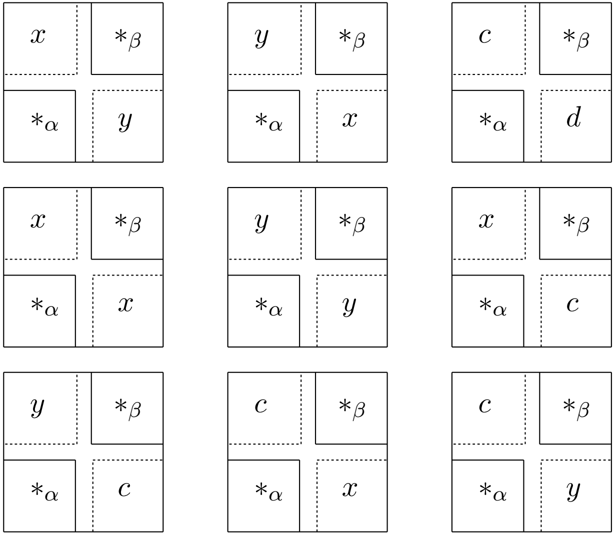
Each commutative, associative binary aggregation function with the **continuous diagonal and continuous Archimedean components** can be expressed as a z**-ordinal** sum of a countable number of semigroups related to **continuous Archimedean t-norms and t-conorms, representable uninorms** and a possibly uncountable number of **trivial semigroups**.

Non-commutative associative aggregation functions



Non-commutative ordinal sum

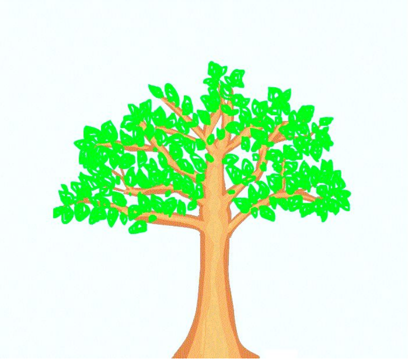


Achievements and future work

- Commutative
  - Characterization of uninorms with continuous underlying functions
  - Characterization of n-uninorms with continuous underlying functions
  - Characterization of CAM aggregation functions continuous around the diagonal
- Non-commutative
  - Characterization of pseudo-uninorms with continuous underlying functions
  - Characterization of pseudo-n-uninorms with continuous underlying functions
  - Characterization of associative aggregation functions continuous around the diagonal
- Corresponding results on general bounded lattices

Idempotent

General



Thank you very much for your attention.