Recent contributions to the theory of differential, difference, and dynamic equations

A talk in the occassion of 65th anniversary of the Mathematical Institute SAS

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Personal anniversary – 10 years of research

Education

Study programme: *Computer modeling* at Technical University of Košice

- BSc: 2014
- MSc: 2016
- PhD: 2020

Work:

- 2020-2021: *Assistent Professor* at Dept. of Mathematics and Theoretical Informatics, Faculty of Electrical Engineering and Informatics, Technical University of Košice
- 2021-current: *Research Fellow* at Mathematical Institute SAS

Research in differential equations (DEs) – brief

- DEs have long played important roles in mathematical modeling.
- The time scale and space structure of the problem determine the final model form.
- Strong application prospect behind the theory of DEs.

Research in differential equations (DEs) – less brief

Subject of interest: functional differential equations (FDEs)

- the unknown function and some of its derivatives are connected for, in general, *different argument values*
- the evolution rate of the processes depends on its past history (or the future)
- A simple classification of FDEs: delay, advanced, neutral DEs

Qualitative (oscillation) theory of FDEs

Many even simple DEs cannot be solved explicitly

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qualitative properties of the solutions deduced from known quantities

$\downarrow$

existence of (oscillatory/nonoscillatory) solutions

asymptotic properties of solutions (boundedness, convergence, monotonic properties)

oscillatory properties of solutions (laws of zeros destribution, oscillation)

Research in the qualitative theory of FDEs

- Qualitative phenomena not relevant for ordinary differential equations
- Heavy restrictions on the deviating argument/proof methods in existing works eliminated (at least partially) its influence

Task: to study the influence of the argument deviation on the qualitative behavior of solutions

Research in the qualitative theory of FDEs

Rules:

- to relax the restrictive assumptions posed on the deviating argument
- do not neglect (even partially) the influence of the deviating argument on the qualitative properties of solutions.

Methods:

- a variety of integral inequalities and iterative techniques
- variational methods
- transformation, factorization and linearization techniques
- comparison principles and integral averaging techniques
- Riccati inequality technique
- fixed point theorems, . . .

Governing equations: First-order FDEs

- the simplest linear DDEs/ADEs

$$x'(t) + q(t)x(\tau(t)) = 0$$

  and NDEs

$$(x(t) + p(t)x(\sigma(t)))' + q(t)x(\tau(t)) = 0$$

- benchmark models in population dynamics and disease transmission models

Governing equations: First-order FDEs

For the DDE

$$x'(t) + q(t)x(\tau(t)) = 0$$

with

$$q, \tau \in C(\mathbb{R}_+), \quad q(t) \geq 0, \quad \tau(t) \leq t, \quad \textit{and} \lim_{t\to\infty} \tau(t) = \infty :$$

ťhe "lower bound" sufficient oscillation condition is

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s)\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{C_1}$$

and the "upper bound" sufficient oscillation condition is

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} q(s)\mathrm{d}s > 1, \quad (\textit{for } \tau'(t) \geq 0). \tag{C_2}$$

- Obvious gap beween (C_1) and (C_2) when $\lim_{t\to\infty} \int_{\tau(t)}^{t} q(s)\mathrm{d}s$ does not exist.
- There is no such $A > 0$: $\limsup_{t\to\infty} \int_{\tau(t)}^{t} q(s)\mathrm{d}s > A$ implies oscillation in case of general delay argument.

Governing equations: First-order FDEs

- For the simplest linear DDEs/ADEs

$$x'(t) + q(t)x(\tau(t)) = 0$$

and NDEs

$$(x(t) + p(t)x(\sigma(t)))' + q(t)x(\tau(t)) = 0$$

we study the problem of finding sharp oscillation constants or sharp (upper and lower) bounds for positive solutions if they exist

- collaboration: G.E. Chatzarakis (Ioannina University, Greece), E. Attia (Damietta University, Egypt)

Governing equations: Second-order FDEs

- Emden-Fowler DDEs/ADEs:

$$\left(r\Phi_\alpha(x')\right)'(t) + q(t)\Phi_\beta(x(\tau(t))) = 0$$

where $\Phi_\alpha(x)$ is Φ_α-Laplacian operator:

$$\Phi_\alpha(x) := |x|^{\alpha-1}x \quad (\alpha > 0)$$

and NDEs

$$\left(r\Phi_\alpha([x(t) + p(t)x(\sigma(t))]')\right)'(t) + q(t)\Phi_\beta(x(\tau(t))) = 0$$

Governing equations: Second-order FDEs

Second-order Emden-Fowler FDEs can be seen as a generalizations of:

- linear ODEs:

$$x''(t) + q(t)x(t) = 0 \quad \left[(rx')'(t) + q(t)x(t) = 0\right]$$

- half-linear ODEs:

$$\left(r\Phi_\alpha(x')\right)'(t) + q(t)\Phi_\alpha(x(t)) = 0 \quad \left(\Phi_\alpha(x) := |x|^{\alpha-1}x\right)$$

- Emden-Fowler ODEs:

$$\left(r\Phi_\alpha(x')\right)'(t) + q(t)\Phi_\beta(x(t)) = 0$$

- linear DDEs/ADEs

$$x''(t) + q(t)x(\tau(t)) = 0$$

- half-linear DDEs/ADEs:

$$\left(r\Phi_\alpha(x')\right)'(t) + q(t)\Phi_\alpha(x(\tau(t))) = 0 \quad \left(\Phi_\alpha(x) := |x|^{\alpha-1}x\right)$$

Governing equations: Second-order FDEs

- Emden-Fowler DDEs/ADEs:

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and NDEs

$$\left(r\Phi_\alpha([x(t) + p(t)x(\sigma(t))]')\right)'(t) + q(t)\Phi_\beta(x(\tau(t))) = 0$$

Governing equations: Second-order FDEs

- a variety of existing methods is based on comparison principles
- effectiveness of the methods is tested on specific equations whose solution spaces are known

Governing equations: Second-order FDEs

Consider the linear ODE

$$x''(t) + q(t)x(t) = 0, \tag{E_l}$$

where

$$q \in C(\mathbb{R}), \quad q(t) \geq 0.$$

Euler differential equation

$$x''(t) + \frac{q_0}{t^2}x(t) = 0, \quad q_0 > 0,$$

$$\text{oscillation} \quad \iff \quad q_0 > \max\{m(1-m) : 0 < m < 1\} = \frac{1}{4}$$

Governing equations: Second-order FDEs

Consider the linear ODE

$$x''(t) + q(t)x(t) = 0, \tag{E_l}$$

where

$$q \in C(\mathbb{R}), \quad q(t) \geq 0.$$

Theorem (Kneser oscillation/nonoscillation theorem)

Eq. (E_l) is oscillatory if

$$\liminf_{t\to\infty} t^2 q(t) > \frac{1}{4}$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^2 q(t) < \frac{1}{4}.$$

Governing equations: Second-order FDEs

Consider the half-linear ODE

$$\left(\Phi_\alpha(x')\right)'(t) + q(t)\Phi_\alpha(x(t)) = 0 \qquad (E_l)$$

where

$$\Phi_\alpha(x) := |x|^{\alpha-1}x \quad (\alpha > 0), \quad q \in C(\mathbb{R}), \quad q(t) \geq 0$$

Euler half-linear differential equation

$$\left(\Phi_\alpha(x')\right)'(t) + \frac{q_0}{t^{\alpha+1}}\Phi_\alpha(x(t)) = 0, \quad q_0 > 0$$

is oscillatory if and only if

$$c(m) := \alpha m^\alpha(1-m) = q_0$$

has no real roots what happens if

$$q_0 > \max\{c(m) : 0 < m < 1\} = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

Governing equations: Second-order FDEs

Consider the half-linear ODE

$$\left(\Phi_\alpha(x')\right)'(t) + q(t)\Phi_\alpha(x(t)) = 0 \qquad (E_l)$$

where

$$\Phi_\alpha(x) := |x|^{\alpha-1}x \quad (\alpha > 0), \quad q \in C(\mathbb{R}), \quad q(t) \geq 0$$

Theorem ((Hille-)Kneser oscillation/nonoscillation theorem)

Eq. (E_l) is oscillatory if

$$\liminf_{t\to\infty} t^{\alpha+1}q(t) > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^{\alpha+1}q(t) < \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

Governing equations: Second-order FDEs

- numerous gaps exist in the qualitative theory, caused by (partial) neglect of the influence of the argument deviation to the qualitative properties of solutions
- problem of finding sharp oscillation constants or sharp (upper and lower) bounds for positive solutions if they exist [open problem even in the linear case]

Governing equations: Second-order FDEs

Second-order half-linear FDE

$$\left(\Phi_\alpha(x')\right)'(t) + q(t)\Phi_\alpha(x(\tau(t))) = 0, \qquad (E)$$

where

$$\Phi_\alpha(x) := |x|^{\alpha-1}x \quad (\alpha > 0), \quad q, \tau \in C(\mathbb{R}_+), \quad q(t) \geq 0, \quad \lim_{t\to\infty} \tau(t) = \infty.$$

$$\beta_* := \begin{cases} \frac{1}{\alpha} \liminf_{t\to\infty} q(t)t\tau^\alpha(t) & \text{for } \tau(t) \leq t, \\ \frac{1}{\alpha} \liminf_{t\to\infty} q(t)t^{\alpha+1} & \text{for } \tau(t) \geq t, \end{cases}$$

$$\lambda_* := \begin{cases} \liminf_{t\to\infty} \frac{t}{\tau(t)} & \text{for } \tau(t) \leq t, \\ \liminf_{t\to\infty} \frac{\tau(t)}{t} & \text{for } \tau(t) \geq t, \end{cases}$$

Governing equations: Second-order FDEs

Second-order half-linear FDE

$$\left(\Phi_\alpha(x')\right)'(t) + q(t)\Phi_\alpha(x(\tau(t))) = 0, \tag{E}$$

where

$$\Phi_\alpha(x) := |x|^{\alpha-1}x \quad (\alpha > 0), \quad q, \tau \in C(\mathbb{R}_+), \quad q(t) \geq 0, \quad \lim_{t\to\infty} \tau(t) = \infty.$$

The method idea:

$$ax > r^{1/\alpha}x'R \quad \text{and} \quad bx < r^{1/\alpha}x'R,$$

$$\Downarrow$$

$$\frac{x}{R^a}\downarrow \quad \text{and} \quad \frac{x}{R^b}\uparrow,$$

Governing equations: Second-order FDEs

Second-order half-linear FDE

$$\left(\Phi_\alpha(x')\right)'(t) + q(t)\Phi_\alpha(x(\tau(t))) = 0, \tag{E}$$

where

$$\Phi_\alpha(x) := |x|^{\alpha-1}x \quad (\alpha > 0), \quad q, \tau \in C(\mathbb{R}_+), \quad q(t) \geq 0, \quad \lim_{t\to\infty} \tau(t) = \infty.$$

Theorem (Kneser oscillation theorem)

(E) is oscillatory if

$$\beta_* > \begin{cases} 0 & \text{for } \lambda_* = \infty, \\ c(\alpha, \lambda_*) & \text{for } \lambda_* < \infty, \end{cases}$$

where

$$c(\alpha, \lambda_*) := \begin{cases} \max\left\{m(1-m)^\alpha \lambda_*^{-\alpha m} : 0 < m < 1\right\} & \text{for } \tau(t) \leq t, \\ \max\left\{m^\alpha(1-m) \lambda_*^{-\alpha m} : 0 < m < 1\right\} & \text{for } \tau(t) \geq t. \end{cases}$$

Governing equations: Second-order FDEs

- collaboration with J. Džurina (Technical University of Košice, Slovakia), S.R. Grace (Cairo University, Egypt), M. Bohner (University of Rolla, USA), J. Graef (University of Tennessee, USA), Tongxing Li (Shandong University, China), A. Zafer (American University of the Middle East, Turkey)

Governing equations: Higher-order FDEs

Linear FDEs

$$L_n x(t) + q(t)x(\tau(t)) = 0$$

where

$$L_n x = \frac{\mathrm{d}}{\mathrm{d}t} r_{n-1}(t) \frac{\mathrm{d}}{\mathrm{d}t} r_{n-2}(t) \cdots \frac{\mathrm{d}}{\mathrm{d}t} r_1(t) \frac{\mathrm{d}}{\mathrm{d}t} x$$

Nonlinear FDEs

$$L_n x(t) + q(t)\Phi_\alpha(x(\tau(t))) = 0$$

- Classification of positive solutions as $t \to \infty$
- Extension of methods proposed for first and second-order FDEs

Extension - Time scales theory

Analogous results for functional difference equations ?

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Time scales theory: unification of discrete and continuous analysis

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Operator $^{\Delta}$ determines dynamic equations on

a time scale $\mathbb{T}$ (closed and nonempty subset of reals)

- $\mathbb{T} = \mathbb{R}$: differential equations
- $\mathbb{T} = \mathbb{Z}$: difference equations
- $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0, q > 1\}$: quantum equations
- collaboration with M. Bohner (University of Rolla, USA), S.R. Grace (Cairo University, Egypt), J. Graef (University of Tennessee, USA), Tongxing Li (Shandong University, China), Ahmed Saied (MI, SAS, Slovakia)

PDEs modeling - Chemotaxis/haptotaxis

- The ability to migrate in response to external signals is shared by many cell populations.
- *Chemotaxis* and *haptotaxis* are biological processes where cells directly move in response to external stimuli.
  - Chemotaxis: cell movement toward or away from chemical gradients
  - Haptotaxis: cell movement along gradients of immobilized molecules in the extracellular matrix

PDEs modeling - Chemotaxis/haptotaxis

Keller-Segel chemotaxis model

$$
\begin{aligned}
u_t &= \nabla \cdot \left(D_u \nabla u - \chi u \nabla v \right), \quad & x \in \Omega, \quad & t > 0, \\
v_t &= D_v \Delta v + \alpha u - \beta v, \quad & x \in \Omega, \quad & t > 0
\end{aligned}
$$

PDEs modeling - Chemotaxis/haptotaxis

Attraction-repulsion chemotaxis model

$$\begin{aligned} u_t &= \nabla \cdot (D_u \nabla u - \chi u \nabla v + \xi u \nabla w), & x \in \Omega, \quad t > 0, \\ v_t &= D_v \Delta v + \alpha_1 u - \beta_1 v, & x \in \Omega, \quad t > 0, \\ w_t &= D_w \Delta w + \alpha_2 u - \beta_2 w, & x \in \Omega, \quad t > 0. \end{aligned}$$

PDEs modeling - Chemotaxis/haptotaxis

Extensions

- logistic source
- nonlinearities in diffusion, attraction, repulsion terms
- gradient dependent flux limitation
- signal consumption
- nonlinear production rates
- two species model

Thanks for the attention