Actions of finite groups on orientable surfaces

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Characterization

Definition

Let $\mathcal{S}_g$ be an orientable surface of genus g, let $\mathrm{Hom}^+ \mathcal{S}_g$ be its group of orientation-preserving homeomorphisms and let G be a finite group.
We say that G acts orientably on $\mathcal{S}_g$ if there is an monomorphism $\varepsilon\colon\ G \to \mathrm{Hom}^+ \mathcal{S}_g$.

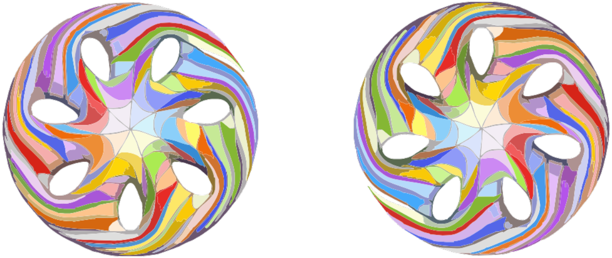
Group G determines a particular symmetry of $\mathcal{S}_g$.

Classification

Definition

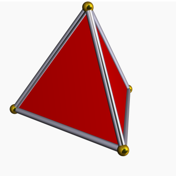
Let $\varepsilon_1 \colon G \to \mathrm{Hom}^+ \mathcal{S}_g$ and $\varepsilon_2 \colon G \to \mathrm{Hom}^+ \mathcal{S}_g$ be two actions of G on $\mathcal{S}_g$. We say that ε_1 and ε_2 are topologically equivalent if there is $\omega \in \mathrm{Aut}\ G$ and $h \in \mathrm{Hom}^+ \mathcal{S}_g$ such that

$$\varepsilon_2(g) = h\varepsilon_1(\omega(g))h^{-1}, \quad \text{for } g \in G.$$

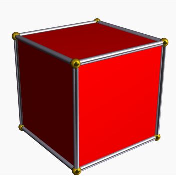


The two actions are essentially the same.

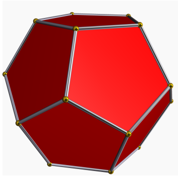
Identification: The sphere $\mathcal{S}_0$



Tetrahedron
A_4

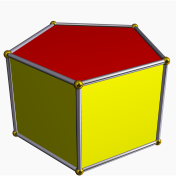


Cube
S_4



Dodecahedron
A_5

Cycles
C_m



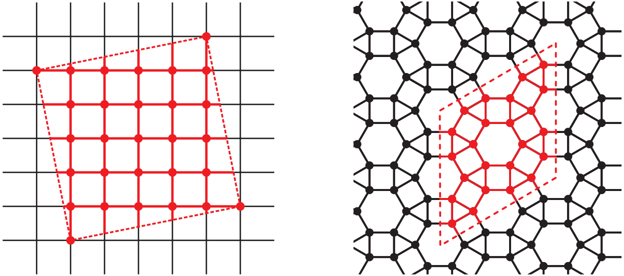
Prisms
D_n

- The discovery of regular solids is attributed to Pythagoras (ca 530 BC);
- We just added infinite families of cycles and prisms (antiprisms);
- Note that (some) duals are missing: octahedron, icosahedron – they are not interesting for us.

Pictures by WikiMedia

Identification: The torus $\mathcal{S}_1$

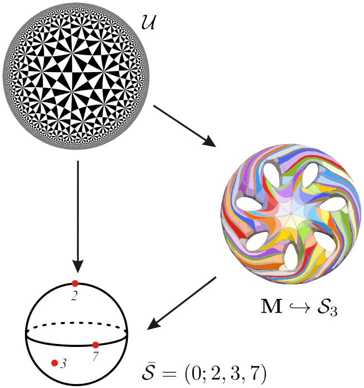
- There are 17 infinite families of finite groups (quotients of wallpaper groups);
- The group G is given by *a pattern* and *two free integral parameters* defining the order of the group;
- See papers by V. Proulx and by O. Šuch.



Identification: Surfaces of higher genera

Proposition (Hurwitz bound)

For $g > 1$, $|G| \leq 84(g-1)$; a list of group actions on $\mathcal{S}_g$ is finite.



For every genus $g \geq 2$ there are finitely many actions of finite groups on $\mathcal{S}_g$.

Riemann existence theorem

Theorem

Every action of G on $\mathcal{S}_g$ can be constructed by means of a pair of Fuchsian groups $K \lhd \Gamma < \mathrm{PSL}(2,\mathbb{R})$ acting discontinuously on the upper-half plane $\mathcal{U}$ and by an order-preserving epimorphism $\eta\colon \Gamma \to G$ with kernel K, where K is a surface group.

- In the representation of the action $(G, \mathcal{S}_g)$ we identify $G = \Gamma/K$ and $\mathcal{S}_g = \mathcal{U}/K$;
- Advantage of such a representation is that we replace topological language by geometry, more precisely group theory!
- In particular, $\mathcal{S}_g$ is a Riemann surface and $\varepsilon(G)$ becomes a group of geometric transformations of $\mathcal{S}_g$.

The state-of-art

$g = 2, 3$	S. A. Broughton, 1990, J. Pure and Applied Algebra;
$g = 4$	O. V. Bogopolskij, 1996, Voprosy Algebry i Logiky;
$g = 4$	H. Kimura, 2003, J. Algebra;
$g = 5$	H. Kimura and A. Kuribayshi 1990, J.Algebra;
$g \leq 9$	KNS, 2024, J. Pure and Applied Algebra; a complete classification;
$g \leq 101$	M. Conder, 2011: "*large groups*", $\lvert G\rvert > 4(g-1)$;
$g \leq 48$	Breuer 2000, a classification with respect to *linear equivalence*;
$g \leq 48$	Paulhus 2019, a database of actions up to linear equivalence.

Note: Large groups have restricted *signatures* $(0; m_1, m_2, m_3)$ or $(0; m_1, m_2, m_3, m_4)$.

Fuchsian group(s)

Each action of a finite group G on $\mathcal{S}_g$ is associated with a Fuchsian group Γ with presentation

$$\langle x_1, x_2, \ldots, x_r, a_1, b_1, \ldots, a_\gamma, b_\gamma \mid x_1^{m_1} = \cdots = x_r^{m_r} = 1, \prod_{i=1}^{\gamma}[a_i, b_i] \prod_{j=1}^{r} x_j = 1 \rangle,$$

where $1 < m_1 \leq m_2 \leq \ldots m_r \in \mathbb{Z}$ and $\gamma \leq g$ is the genus of the quotient surface $\mathcal{S}_g/G$ and m_i are the branch indices of the (regular) branched covering $\mathcal{S}_g \to \mathcal{S}_g/G \cong \mathcal{S}_\gamma$.

The group Γ has two different interpretations:
(a) it is the fundamental group of the quotient orbifold $\mathcal{S}_g/G$,
(b) Γ acts as a group of symmetries of the universal cover.

Definition

The signature $(\gamma; m_1, m_2, \ldots, m_r)$ completely determines Γ.

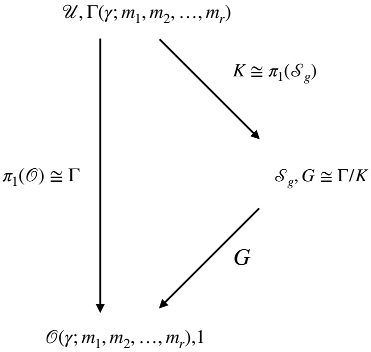
Riemann-Hurwitz equation

The numerical parameters of a signature are given by Riemann-Hurwitz equation:

$$2-2g=|G|\left(2-2\gamma-\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right).$$

- $1 \leq |G| \leq 84(g-1)$;
- $0 \leq \gamma < g$;
- m_i is a nontrivial divisor of $|G|$;
- $0 \leq r \leq 2\gamma+2$.

Action of a group on a surface, Koebe theorem



- The group G has action on the surface $\mathcal{S}_g$;
- There exist the epimorphism $\xi : F \to G$;
- $\ker \xi$ is torsion-free $\Rightarrow$ ξ preserves orders of elements of Γ;
- The quotient surface $\mathcal{S}_g / G = (\gamma; m_1, m_2, \ldots, m_r)$ is an orbifold.

Generating all discrete groups of given genus

Input: The genus, $g > 1$,
Output: A list of actions given by epimorphisms $\Gamma \to G$.

Step 1: Find all numerical solutions of Riemann-Hurwitz equation with respect to $\langle(\gamma; m_1, m_2, \ldots, m_r), |G|\rangle$.

Step 2: Generate all order-preserving epimorphisms $\Gamma \to G$. The group G is chosen from Small Group Library, if possible. Otherwise, one can use coset enumeration to find normal subgroups of Γ, of index $|G|$.

Remark

Not all numerical solutions have realisations. For instance, $|G| = 84(g-1)$, $\gamma = 0$, $(m_1, m_2, m_3) = (2, 3, 7)$ *is numerically admissible solution for every* $g > 1$, *however, for* $g = 2$ *and infinitely many other genera there is no discrete group acting with the above parameters.*

Action of $\mathbb{Z}_2$ on $\mathcal{S}_2$

The epimorphisms $\Gamma \to G$ are encoded as vectors of images of generators of Γ in $\xi\colon \Gamma \to \Gamma/K \cong G$

$$\xi := (\xi(a_1), \xi(b_1), \dots, \xi(a_\gamma), \xi(b_\gamma), \xi(x_1), \dots, \xi(x_r)).$$

We have an epimorphism $\xi\colon \Gamma(1; 2, 2) \to (\mathbb{Z}_2, +)$, encoded by a vector $(1, 0; 1, 1)$, i.e. $a \mapsto 1, b \mapsto 0, x_1 \mapsto 1, x_2 \mapsto 1$, but the set of all *smooth epimorphisms* $\Gamma \to G$ is

$$\mathrm{Epi}_0\,(\Gamma(1; 2, 2), \mathbb{Z}_2) = \{(0, 0; 1, 1), (0, 1; 1, 1), (1, 0; 1, 1), (1, 1; 1, 1)\}.$$

- How many equivalence classes are there?
- By Broughton's list of actions of genera 2, there is only one class.
- Since $\mathrm{Aut}\ G$ is trivial, there must be another reason for one class. A nontrivial (transitive) action of $\mathrm{Aut}^+\,\Gamma$ on $\mathrm{Epi}_0\,\Gamma, \mathbb{Z}_2$...?

Equivalence of actions

Observation (Necessary condition)

If two actions G_1 *and* G_2 *on* $\mathcal{S}_g$ *are equivalent, then both* $G_1 \cong G_2$ *and* $\Gamma_1 \cong \Gamma_2$.

Problem (Topological equivalence)

Given a (finite) set of vectors $\mathrm{Epi}_0(\Gamma, G)$, *find the classes of topological equivalence.*

Theorem (Lloyd 1972)

Two order preserving epimorphisms $\xi_1, \xi_2 \in \mathrm{Epi}_0(\Gamma, G)$ *are equivalent if and only if there exists* $a \in \mathrm{Aut}\ G$ *and* $\alpha \in \mathrm{Aut}^+\,\Gamma$ *such that* $\xi_2 = a\xi_1\alpha$.

Mapping class group problem

- To identify the equivalence classes, we have to understand the action of an infinite group $\mathrm{Aut}^+\,\Gamma$ on $\mathrm{Epi}_0(\Gamma, G)$;
- Unfortunately, we do not know how to find a suitable generating set for $\mathrm{Aut}^+\,\Gamma$ for a general Fuchsian group!
- In fact, the essential part of the problem is to understand the action of $\mathrm{Out}\,\Gamma = \mathrm{Aut}^+\,\Gamma/\,\mathrm{Inn}\,\Gamma$ - a mapping class group;

Fuchsian groups are ”pretty close” to free groups:

- Zieshang (1966) proved that every automorphism in $\mathrm{Aut}\,\Gamma$ lifts to an automorphism of the corresponding free group $F_{r+2\gamma}$;
- Birman (1974) and Chow (1948) proved that subgroup of $\mathrm{Aut}\,\Gamma$ permuting the elements of finite order has the structure of *braid group*;
- McCool (1996) derived the complete generating set of $\mathrm{Aut}\,\Gamma$ for *surface groups*.

Action of $\mathrm{Aut}^+\,\Gamma$ on $\mathrm{Epi}_0(\Gamma, G)$

A planar Fuchsian group is $\Gamma = \langle x_1, x_2, \ldots x_r \mid x_i^{m_i} = 1, i = 1, \ldots, r, x_1 x_2 \ldots x_r = 1\rangle$.

The **vertical action** is the action of the image of the pure braid group. It is generated by the images of the "pure braid group" generators $\langle \bar{A}_{s,t}\rangle$, $1 \leq s < t \leq r$ and the action on X is defined by

$$(y_1, \ldots, y_r) \mapsto (\mathcal{A}_{s,t}(y_1), \ldots, \mathcal{A}_{s,t}(y_r)), \qquad 1 \leq s < t \leq r.$$

The **horisontal action** of $\mathrm{Aut}^+\,\Gamma$ on X is generated by the $\mathcal{P}$-invariant generators $\bar{c}_i$ ($i \approx i+1$),

$$(y_1, \ldots, y_i, y_{i+1}, \ldots, y_r) \mapsto (y_1, \ldots, y_i y_{i+1} y_i^{-1}, y_i, \ldots, y_r).$$

Generators of pure braid group

- The generators of $\operatorname{Aut}^+\Gamma$ are the generators $\bar{A}_{s,t} = \mathcal{A}_{s,t}$ of the image P_r^* of pure braid group and the images $\bar{c}_i$ such that $i \approx i+1$ in the equivalence defined by partition of elements of finite order given by their orders.
- $c_i : x_i \mapsto x_i x_{i+1} x_i^{-1}$, $x_{i+1} \mapsto x_i$ and $x_j \mapsto x_j$ for $j \notin \{i, i+1\}$.
- Explicit definition of $\mathcal{A}_{s,t}$ has been derived from the paper Tap (1988)

$$\mathcal{A}_{s,t}(x_i) = \begin{cases} x_i, & t < i \text{ or } i < s, \\ x_i^{x_s}, & t = i, \\ x_i^{x_s x_t}, & s = i, \\ x_i^{[x_s, x_t]}, & s < i < t \end{cases}$$

for $1 \leq s < t \leq r$

Special cases

- If G is abelian, the equivalence is determined by the action of $\mathrm{Sym}(r, \mathcal{P})$ on the branch indices.
- If $m_1 = m_2 = \cdots = m_r$, then $\mathcal{P}$ has just one class and the equivalence for abelian G is determined by the *horisontal action*.
- If $m_1 < m_2 < \cdots < m_r$, then $\mathcal{P}$ is "=" and the equivalence is determined by the *vertical action* of P_r^*.
- If $r = 3$, then $P^* \cong \mathrm{Inn}\,\Gamma$. In particular, the classification of such actions with respect to the equivalence coincides with the classification of oriented hypermaps of given genus up to duality. (see Conder's lists up to genus 301).

Surface groups

- Surface group is Fuchsian group with signature $(\gamma; -)$,
- $\Gamma(\gamma; -) \cong \pi(\mathcal{S}_\gamma)$ is isomorphic to the fundamental group of the surface,
- If G acts on $\mathcal{S}_g$ with signature $(\gamma; -)$ then the action is regular.

McCool (1996) derived the following generating set of $\mathrm{Aut}^+ \Gamma$:

Automorphisms $\mathcal{C}_j$:

$$\begin{aligned}
&\mathcal{C}_1 \colon a_1 \mapsto a_1 b_1^{-1} \\
&\mathcal{C}_2 \colon b_1 \mapsto b_1 a_1 \\
&\mathcal{C}_3 \colon a_1 \mapsto a_1 b_1^{-1} a_2 b_2 a_2^{-1}, \quad a_2 \mapsto a_2 b_2^{-1} a_2 b_1 a_2, \\
&\qquad b_1 \mapsto a_2 b_2^{-1} a_2^{-1} b_1 a_2 b_2 a_2^{-1} \\
&\mathcal{C}_4 \colon a_i \mapsto a_{i+1} \quad b_i \mapsto b_{i+1}
\end{aligned}$$

where

$$\Gamma = \langle a_1, b_1, \ldots, a_\gamma, b_\gamma \mid \prod_{j=1}^{\gamma} [a_j, b_j] = 1 \rangle.$$

Where are we?

We have the classification of group actions for surfaces of genera $2 \leq g \leq 9$.

J. Karabáš, R. Nedela, and M. Skyvová, *Computing equivalence classes of finite group actions on orientable surfaces*, J. Pure and Appl. Algebra 228, January 2024.

https://www.savbb.sk/~karabas/science/discactions.html.

Genus	adm	H1	top
2	21	21	21
3	49	55	55
4	64	73	75
5	93	116	118
6	87	105	108
7	148	208	210
8	108	141	150
9	270!	428?	441

Where are we going?

Problem 1: **Implement *effective algorithm* computing the equivalence classes of discrete groups.** It requires to identify a finite set of generators of $\mathrm{Aut}^+ \Gamma$ for a general case.

Problem 2: **Enumerate of (or generate) actions of certain types of groups.** Small abelian groups have huge number of actions. It will be useful to enumerate them, or at least find an independent way how to generate them, a recursive formula?

Notes

- A formula counting $|\mathrm{Epi}_0(\Gamma, \mathbb{Z}_n)|$ was derived by Mednykh and N. (2006).
- It is a multivariable arithmetic function and we have derived its aditive form.
- A multiplicative form was derived by Liskovets (2009).
- It has applications in map enumeration, in enumeration of coverings between manifolds and bit surprisingly, in 2-dimensional gravity models in theoretical physics.

Thank you for your attention!