Discrete probability distributions generated by partial summations

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Outline

Partial summations - definition

$$P_x = \sum_{j=x}^{\infty} g(j) P_j^*, \qquad x = 0, 1, 2, \ldots$$

- $\{P_j^*\}_{j=0}^{\infty}$ - discrete probability distribution (parent)
- $\{P_x\}_{x=0}^{\infty}$ - discrete probability distribution (descendant)
- $g(j)$ - real function (which specifies the partial summation)
- some special cases mentioned in the monograph Univariate Discrete Distributions (Johnson, N.L., Kemp, A.W, Kotz, S., 2005, Wiley)

Partial summations - more in detail

$$
\begin{aligned}
P_0 &= g(0)P_0^* + g(1)P_1^* + g(2)P_2^* + \dots \\
P_1 &= \qquad\qquad\quad g(1)P_1^* + g(2)P_2^* + \dots \\
P_2 &= \qquad\qquad\qquad\qquad\quad g(2)P_2^* + \dots \\
&\vdots
\end{aligned}
$$

Partial summations - history

- some special cases have real-world motivations
  - $g(j) = c$ - reliability theory
  - $g(j) = c/j$ - economy (income underreporting), management (ideal time to order new supplies in a shop)
- from purely mathematical point of view, in the (not so distant) past partial summations were seen as a tool for creating new distributions and/or establishing links between distributions
- relations between properties (moments, probability generating functions) of parents and descendants derived[1]

[1] Mačutek, J. (2003), On two types of partial summations. *Tatra Mountains Mathematical Publications* 26, 403-410.

Partial summations - present

- any two discrete distributions defined on the same support are connected by partial summation[2]
- this partial summation is uniquely determined

[2] Wimmer, G., Mačutek, J. (2012). New integrated view at partial-sums distributions. *Tatra Mountains Mathematical Publications* 51, 183-190.

Invariance with respect to partial summations

- for all discrete distributions there is $g(j)$ such that the parent and the descendant distributions are the same, i.e., $P_x^* = P_x$ for $x = 0, 1, 2, \ldots$
- hence, the distribution $\{P_j^*\}_{j=0}^{\infty}$ is invariant with respect to the partial summation given by $g(j)$
- function $g(j)$ is uniquely determined
- function $g(j)$ is a (new) characteristic of discrete distributions

Invariance - examples

Poisson distribution

$$P_x^* = \sum_{j=x}^{\infty} \frac{j - \lambda + 1}{j + 1} P_j^*, \qquad x = 0, 1, 2, \ldots$$

geometric distribution[3]

$$P_x^* = \sum_{j=x}^{\infty} p P_j^*, \qquad x = 0, 1, 2, \ldots$$

[3] Wimmer, G., Kalas, J. (1999). A characterization of the geometric distribution. *Tatra Mountains Mathematical Publications* 17, 325-329.

Invariance - general[4]

$$g(j) = 1 - \frac{P^*_{j+1}}{P^*_j}, \qquad j = 0, 1, 2, \ldots$$

[4] Mačutek, J. (2003). On two types of partial summations. *Tatra Mountains Mathematical Publications* 26, 403-410.

Partial summations - summary

- the world of discrete distributions seen from a new point of view
- new properties of distributions and relations among them appear which are difficult (impossible?) to notice from 'old' perspectives

Partial summations - open problem no. 1

- for every pair of distributions P_x and P_x^* there exists function $g(j)$ such that

$$P_x = \sum_{j=x}^{\infty} g(j) P_j^*, \qquad x = 0, 1, 2, \ldots$$

- but function $g(j)$ uniquely determines another distribution (the third one)
- hence, we have a 'family' consisting of three distributions - in fact, many such families
- what exactly is the relation among the three distributions, in which sense are they related?

Repeated partial summations

$$P_x^{(1)} = c_1 \sum_{j=x}^{\infty} g(j) P_j^*, \qquad x = 0, 1, 2, \ldots$$

$$P_x^{(2)} = c_2 \sum_{j=x}^{\infty} g(j) P_j^{(1)}, \qquad x = 0, 1, 2, \ldots$$

$$\vdots$$

$$P_x^{(n)} = c_n \sum_{j=x}^{\infty} g(j) P_j^{(n-1)}, \qquad x = 0, 1, 2, \ldots$$

$$\vdots$$

- we want to find $\lim_{n \to \infty} P_x^{(n)}$, $x = 0, 1, 2, \ldots$, if it exists

Repeated geometric partial summations

$$P_x^{(1)} = c_1 \sum_{j=x}^{\infty} P_j^*, \qquad x = 0, 1, 2, \ldots$$

$$P_x^{(2)} = c_2 \sum_{j=x}^{\infty} P_j^{(1)}, \qquad x = 0, 1, 2, \ldots$$

$$\vdots$$

$$P_x^{(n)} = c_n \sum_{j=x}^{\infty} P_j^{(n-1)}, \qquad x = 0, 1, 2, \ldots$$

$$\vdots$$

Repeated geometric partial summations

Theorem[5]

If

$$\lim_{x\to\infty} \frac{P^*_{x+1}}{P^*_x} = q \in (0,1),$$

then

$$\lim_{n\to\infty} P^{(n)}_x = (1-q)q^x, \qquad x = 0, 1, 2, \ldots$$

[5] Mačutek, J. (2006). A limit property of the geometric distribution. *Theory of Probability and Its Application* 50, 316-319.

Repeated geometric partial summations

Theorem[5]

If

$$\lim_{x\to\infty} \frac{P^*_{x+1}}{P^*_x} = q \in (0,1),$$

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$$\lim_{n\to\infty} P^{(n)}_x = (1-q)q^x, \qquad x = 0,1,2,\dots$$

- The proof uses probability generating functions

$$G_n(t) = \sum_{j=0}^{n-1} (-1)^j \frac{t^j \prod_{x=0}^{j} c_{n-x}}{(1-t)^{j+1}} + (-1)^n \frac{t^n \prod_{j=1}^{n} c_j}{(1-t)^n} G^*(t)$$

[5] Mačutek, J. (2006). A limit property of the geometric distribution. *Theory of Probability and Its Application* 50, 316-319.

Computational study in R

- parent distribution must have finite support
- we will present results for binomial distribution as the parent
- many different partial summations (i.e., many different functions $g(j)$) applied to the binomial distribution

Salvia-Bolinger partial summation

$$Q_j = \frac{a \prod_{k=1}^{j}(k-a)}{(j+1)!} \quad \Longrightarrow \quad g(j) = 1 - \frac{Q_{j+1}}{Q_j} = \frac{a+1}{j+2}$$

$$P_x^{(n)} = c_n \sum_{j=x}^{\infty} \frac{a+1}{j+2} P_j^{(n-1)}, \qquad x = 0, 1, 2, \ldots$$

$$n = 1, 2, 3, \ldots$$

Salvia-Bolinger partial summation

$$P_x^{(1)} = c_1 \sum_{j=x}^{\infty} \frac{a+1}{j+2} P_j^*, \qquad x = 0, 1, 2, \ldots$$

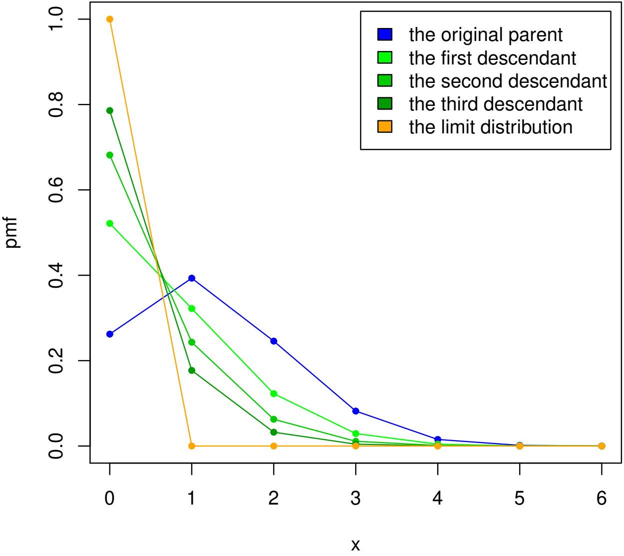
$$P_x^{(2)} = c_2 \sum_{j=x}^{\infty} \frac{a+1}{j+2} P_j^{(1)}, \qquad x = 0, 1, 2, \ldots$$

$$\vdots$$

$$P_x^{(n)} = c_n \sum_{j=x}^{\infty} \frac{a+1}{j+2} P_j^{(n-1)}, \qquad x = 0, 1, 2, \ldots$$

$$\vdots$$

Salvia-Bolinger partial summation



- parent - biniomial $n = 6,\ p = 0.2$
- summation Salvia-Bolinger, $a = 0.5$
- limit deterministic??

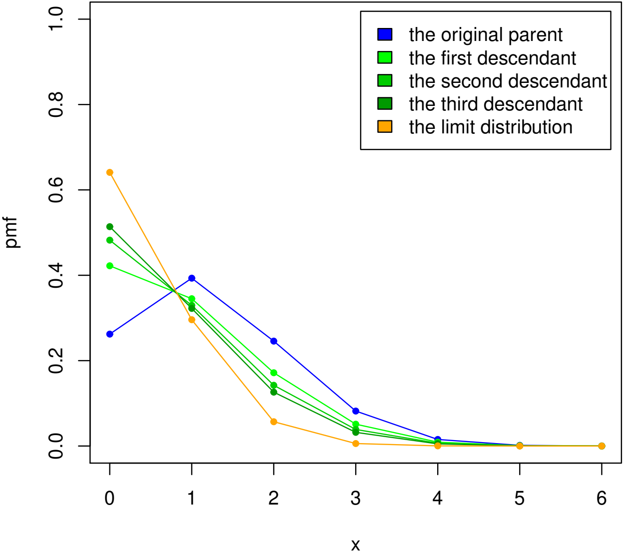
Poisson partial summation

$$Q_j = \frac{e^{-a}a^j}{j!} \quad \Longrightarrow \quad g(j) = 1 - \frac{Q_{j+1}}{Q_j} = \frac{j+1-a}{j+1}$$

$$P_x^{(n)} = c_n \sum_{j=x}^{\infty} \frac{j+1-a}{j+1} P_j^{(n-1)}, \qquad x = 0, 1, 2, \dots$$

$$n = 1, 2, 3, \dots$$

Poisson partial summation



- parent - binomial
  $n = 6, \ p = 0.2$
- summation
  Poisson
  $a = 0.5$
- limit
  not deterministic??

Power method

- let all coordinates of vector $\mathbb{P}^*$ be nonzero
- let matrix A have only one dominant eigenvalue λ_D, let $\lambda_D \in \mathbb{R}$
- let v be the eigenvector associated with eigenvalue λ_D

$$\lim_{n\to\infty} \frac{A^n\mathbb{P}^*}{\|A^n\mathbb{P}^*\|} = v$$

$$\lim_{n\to\infty} \left(A\frac{A^n\mathbb{P}^*}{\|A^n\mathbb{P}^*\|}\right)^\top \left(\frac{A^n\mathbb{P}^*}{\|A^n\mathbb{P}^*\|}\right) = \lambda_D$$

Remarks on power method

- known also as von Mises iteration algorithm
- used by Google search engine
- convergence depends on $\frac{|\lambda_D|}{|\lambda_2|}$ - the higher the ratio, the better
- different norms can be used

Matrix notation of partial-sums distributions

we consider only distributions defined on finite support

$$P_x = \sum_{j=x}^{q-1} g(j) P_j^*, \qquad x = 0, 1, \ldots, q-1 \qquad \Longleftrightarrow \qquad \mathbb{P} = A\mathbb{P}^*$$

$$\mathbb{P}^* = \left(P_0^*, P_1^*, \ldots, P_{q-1}^*\right)^\top \qquad \mathbb{P} = \left(P_0, P_1, \ldots, P_{q-1}\right)^\top$$

$$A = \begin{pmatrix} g(0) & g(1) & \ldots & g(q-1) \\ 0 & g(1) & \ldots & g(q-1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & g(q-1) \end{pmatrix}$$

Matrix notation of repeated partial summations

- neglect for a while normalization constants

$$
\begin{aligned}
\mathbb{P}^{(1)} &= A\mathbb{P}^* \\
\mathbb{P}^{(2)} &= A\mathbb{P}^{(1)} = AA\mathbb{P}^* = A^2\mathbb{P}^* \\
&\vdots \\
\mathbb{P}^{(n)} &= A\mathbb{P}^{(n-1)} = A^n\mathbb{P}^* \\
&\vdots
\end{aligned}
$$

- the n-th descendant has probability mass function $\frac{\mathbb{P}^{(n)}}{\|\mathbb{P}^{(n)}\|_1}$

$$
\mathbb{P}^{(\infty)} = \lim_{n\to\infty} \frac{\mathbb{P}^{(n)}}{\|\mathbb{P}^{(n)}\|_1} = \lim_{n\to\infty} \frac{A^n\mathbb{P}^*}{\|A^n\mathbb{P}^*\|_1}
$$

Relation to power method

Power method

$$\lim_{n\to\infty} \frac{A^n \mathbb{P}^*}{\|A^n \mathbb{P}^*\|_2} = v$$

Repeated partial summations

$$\mathbb{P}^{(\infty)} = \lim_{n\to\infty} \frac{\mathbb{P}^{(n)}}{\|\mathbb{P}^{(n)}\|_1} = \lim_{n\to\infty} \frac{A^n \mathbb{P}^*}{\|A^n \mathbb{P}^*\|_1}$$

$$\Downarrow$$

$$\mathbb{P}^{(\infty)} = \frac{v}{\|v\|_1}$$

How to find limit distribution?

I. find dominant eigenvalue of matrix A

$$A = \begin{pmatrix} g(0) & g(1) & \dots & g(q-1) \\ 0 & g(1) & \dots & g(q-1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g(q-1) \end{pmatrix}$$

$$\lambda_D = \max_{j \in \{0,1,\dots,q-1\}} |\lambda_j| = \max_{j \in \{0,1,\dots,q-1\}} |g(j)|, \quad \text{if it exists}$$

II. find eigenvector v associated with eigenvalue λ_D

III. normalize v so that $\sum v_i = 1$, i.e.,

$$\frac{v}{\|v\|_1}$$

The binomial or deterministic distribution is the limit distribution for $g(j)$ given by all distributions from Katz family (binomial, negative binomial with geometric as its special case, Poisson).

Repeated partial summations in general

I. let the dominant eigenvalue exist,

$$\lambda_D = g(r), \qquad r \in \{0, 1, \dots, q-1\}$$

II. find eigenvector associated with λ_D

$$(A - \lambda_D I)v = 0$$

$$\begin{pmatrix} g(0) - g(r) & g(1) & \dots & g(q-1) \\ 0 & g(1) - g(r) & \dots & g(q-1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g(q-1) - g(r) \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{q-1} \end{pmatrix} = 0$$

Repeated partial summations in general

- solution of the system of equations:

$$v_k = \begin{cases} \frac{t}{(g(r))^k} \prod_{j=1}^{k} (g(r) - g(j-1)), \quad t \in \mathbb{R} & \\ \qquad \text{if } k = 0, 1, \ldots, r & \\ 0 \qquad \text{if } k = r+1, r+2, \ldots, q-1 & \end{cases}$$

Partial summations - open problem no. 2 and 3

- how to find the limit distribution if there is no dominant eigenvalue?
- how to find the limit distribution for parents with infinite support?

Parametrized partial summations

- consider only discrete distributions with one parameter
- recall invariance - for every distribution $P*$ there is function $g(j)$ such that

$$P_x^* = \sum_{j=x}^{\infty} g(j) P_j^*$$

- emphasizing the parameter, one can write

$$P_x^*(a) = \sum_{j=x}^{\infty} g(j, a) P_j^*(a)$$

Parametrized partial summations

- change now the parameter value while keeping the formula for function $g(j)$, i.e., consider summation

$$P_x = \sum_{j=x}^{\infty} g(j, \lambda) P_j^*(a)$$

Parametrized partial summations - Poisson summation

- Poisson distribution with parameter a is invariant with respect to summation

$$P_x^*(a) = \sum_{j=x}^{\infty} \frac{j-a+1}{j+1} P_j^*(a)$$

- consider summation

$$P_x = c \sum_{j=x}^{\infty} \frac{j-\lambda+1}{j+1} \frac{e^{-a}a^j}{j!}$$

- we have

$$P_0 = \frac{a-\lambda+\lambda e^{-a}}{a(a+1-\lambda)}$$

- descendant distribution has two parameters

Parametrized partial summations - geometric summation

- geometric distribution with parameter a is invariant with respect to summation

$$P_x^*(a) = \sum_{j=x}^{\infty} a P_j^*(a)$$

- consider summation

$$P_x = c \sum_{j=x}^{\infty} \lambda a (1-a)^j$$

- in this case, normalization constant c 'cancels' the new parameter λ, which means that parent and descendant are the same.

- distributions are either sensitive (e.g., Poisson) or resistant (e.g., geometric) with respect to a change of parameter value in function $g(j)$
- many distributions are sensitive
- open problem no. 5 - what is the common property of resistant distributions?

Ďakujem za pozornosť