Zero-divisor graphs of posets

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– 2007 Nimbhorkar, Wasadikhar, DeMeyer: Beck's conjecture holds in meet-semilattices with 0.

– 2010 Halaš, Jukl and Halaš, Länger: Beck's conjecture holds for posets (qosets) (with 0).

Extremal graph theory

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In this case $\omega = M$ (antichain = clique) and $\chi = m$, partition of P into m independent sets (chain = independent set).

Given poset $(P, \leq)$ we assign a graph $\Gamma(P) = (P, E)$: its vertices are elements of P, and $p, q \in P$ are connected by an edge iff $L(p, q) = L(P)$, i.e., $pq \in E$ provided the sets of lower bounds of p, q and P are the same.

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Since $p \perp q$ impies $p \,||\, q$, thus zero-divisor graph of P is a subgraph of the incomparability graph.

Zero-divisor graphs of posets – finite case

Theorem

Let $(P, \leq)$ be a partially ordered set such that $\omega(\Gamma(P))$ is finite. Then the zero-divisor graph $\Gamma(P)$ fulfils the Beck's conjecture, i.e., $\omega(\Gamma(P)) = \chi(\Gamma(P))$.

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$v \perp (K \setminus \{v\})$ implies $(K \setminus \{v\}) \cup \{r_x, r_y\}$ is a larger clique.

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$\chi(\Gamma(P)) = \aleph_0$ means that P can be partitioned into countably many independent sets (independent set = no disjoint pair).

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Let κ and $\lambda > 2^\kappa$ be infinite cardinals. If P is a separative poset of cardinality λ, then P cannot be covered by the union of κ independent sets, i.e., $\chi(\Gamma(P)) > \kappa$.

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The proof uses partition calculus, the bound $\lambda > 2^\kappa$ is from Erdős-Rado theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$.

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For any $i, j, k \in S$ we have $r_{i,j}, r_{j,k}, r_{i,k} \in A_t$ for some $t < \kappa$.

However, at least two of these three r are disjoint, hence A_t is not independent (contradiction).

Theorem

Let P be a separative poset satisfying the ccc condition, and let $|P| > 2^\kappa$ for some cardinal $\kappa \geq \aleph_0$. Then $\chi(\Gamma(P)) > \kappa$ and $\omega(\Gamma(P)) = \aleph_0$.

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For sets I, J, let $P(I,J) = \{p \subseteq I \times J : |p| < \aleph_0, p \text{ is a function}\}$ be the set of all finite partial functions from I to J.

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Boolean algebras $\Longleftrightarrow$ Boolean rings