Introduction to the intuitionistic fuzzy sets

Katarína Čunderlíková

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
814 73 Bratislava
cunderlikova.lendelova@gmail.com

Smolenice 2024
65th Anniversary of the Mathematical Institute
Slovak Academy of Sciences

Intuitionistic Fuzzy Sets

K. T. Atanassov in 1983

IF-set $\mathbf{A}$ on a space Ω

$$\mathbf{A} = \{(\mu_A, \nu_A), \mu_A + \nu_A \leq 1_\Omega\}$$

$\mu_A : \Omega \to [0,1]$ - membership function
$\nu_A : \Omega \to [0,1]$ - non membership function
$\pi_A = 1_\Omega - \mu_A - \nu_A > 0$ - degree of uncertainty

IF-event on a measurable space $(\Omega, \mathcal{S})$

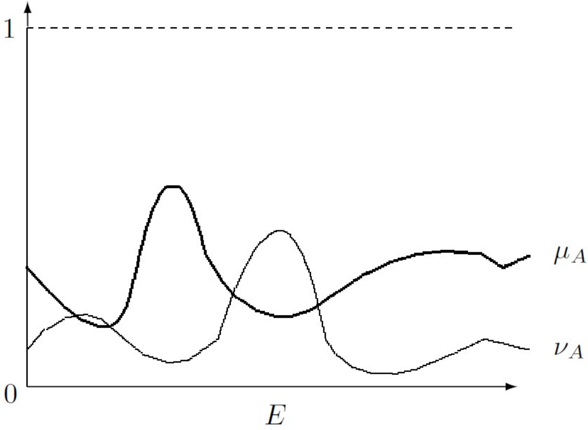
$$\mathbf{A} = (\mu_A, \nu_A),\ \mu_A + \nu_A \leq 1_\Omega$$

$\mu_A, \nu_A : \Omega \to [0,1]$ - $\mathcal{S}$-measurable.

The family of all IF-events will be denoted by $\mathcal{F}$.

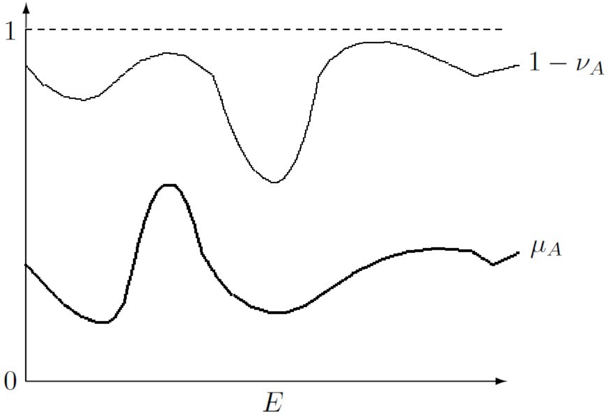
Intuitionistic Fuzzy Sets

First geometrical interpretation



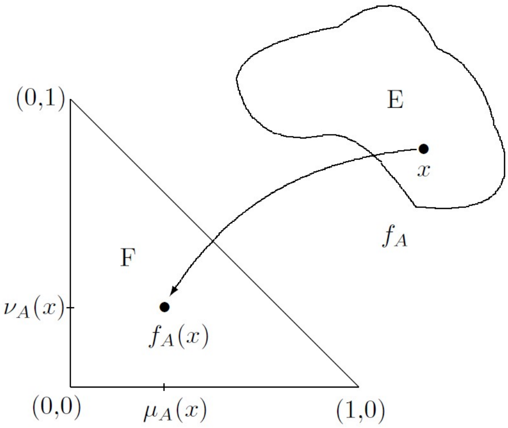
Intuitionistic Fuzzy Sets

Second geometrical interpretation



Intuitionistic Fuzzy Sets

Third geometrical interpretation



Intuitionistic Fuzzy Events

Lukasiewicz binary operations on $\mathcal{F}$

$\mathbf{A} = (\mu_A, \nu_A)$, $\mathbf{B} = (\mu_B, \nu_B)$

$$\begin{aligned} \mathbf{A} \oplus \mathbf{B} &= \left((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0\right) \\ \mathbf{A} \odot \mathbf{B} &= \left((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1\right) \end{aligned}$$

Lattice operations on $\mathcal{F}$

$$\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$$

$$\mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$$

Partial ordering on $\mathcal{F}$

$$\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B$$

Intuitionistic Fuzzy Probability

B. Riečan in 2003

$\mathcal{P} : \mathcal{F} \to \mathcal{J}$

(i) $\mathcal{P}((1_\Omega, 0_\Omega)) = [1, 1]$, $\mathcal{P}((0_\Omega, 1_\Omega)) = [0, 0]$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then

$$\mathcal{P}(\mathbf{A} \oplus \mathbf{B}) = \mathcal{P}(\mathbf{A}) + \mathcal{P}(\mathbf{B})$$

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$, then $\mathcal{P}(\mathbf{A}_n) \nearrow \mathcal{P}(\mathbf{A})$.

$\mathbf{A}_n = (\mu_{A_n}, \nu_{A_n})$, $\mathbf{A} = (\mu_A, \nu_A)$

$$\mathbf{A}_n \nearrow \mathbf{A} \Longleftrightarrow \mu_{A_n} \nearrow \mu_A,\ \nu_{A_n} \searrow \nu_A$$

Intuitionistic Fuzzy State

Theorem

Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ and $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$. Then $\mathcal{P}$ is an IF-probability if and only if $\mathcal{P}^\flat$ and $\mathcal{P}^\sharp$ are IF-states.

$\mathbf{m} : \mathcal{F} \to [0, 1]$

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then

$$\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$$

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$, then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Representations of Intuitionistic Fuzzy State

Butnariu-Klement formulation

To each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists exactly one probability measure $P : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that

$$\mathbf{m}(\mathbf{A}) = (1-\alpha)\int_\Omega \mu_A dP + \alpha\left(1 - \int_\Omega \nu_A dP\right)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

Ciungu-Riečan formulation

To each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exist some probability measures $P, Q : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that

$$\mathbf{m}(\mathbf{A}) = \int_\Omega \mu_A dP + \alpha\left(1 - \int_\Omega (\mu_A + \nu_A) dQ\right)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

MV-algebra of IF-sets

$(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$

- $\mathcal{M} = \{\mathbf{A} = (\mu_A, \nu_A); \mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$ – measurable functions$\}$

Operations

$\mathbf{A} = (\mu_A, \nu_A)$, $\mathbf{B} = (\mu_B, \nu_B)$

$$\begin{aligned} \mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)), \\ \neg \mathbf{A} &= (1_\Omega - \mu_A, 1_\Omega - \nu_A), \end{aligned}$$

where $(\Omega, \mathcal{S})$ be a measurable space, $\mathcal{S}$ be a σ-algebra.

MV-algebra of IF-sets

The corresponding lattice group

$(\mathcal{G}, +, \leq)$

- $\mathcal{G} = \{\mathbf{A} = (\mu_A, \nu_A); \mu_A, \nu_A : \Omega \longrightarrow R$ are $\mathcal{S} -$ measurable functions$\}$

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= (\mu_A + \mu_B, \nu_A + \nu_B - 1_\Omega), \\ \mathbf{A} \leq \mathbf{B} &\iff \mu_A \leq \mu_B, \nu_A \geq \nu_B. \end{aligned}$$

- the neutral element is $\mathbf{0} = (0_\Omega, 1_\Omega)$,

$$\mathbf{A} - \mathbf{B} = (\mu_A - \mu_B, \nu_A - \nu_B + 1_\Omega)$$

- the lattice operations

$$\begin{aligned} \mathbf{A} \vee \mathbf{B} &= (\mu_A \vee \mu_B, \nu_A \wedge \nu_B), \\ \mathbf{A} \wedge \mathbf{B} &= (\mu_A \wedge \mu_B, \nu_A \vee \nu_B). \end{aligned}$$

State on MV-algebra $\mathcal{M}$

$m : \mathcal{M} \to [0,1]$ - monotone mapping

(i) $m((1_\Omega, 0_\Omega)) = 1$, $m((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, then

$$m(\mathbf{A} \oplus \mathbf{B}) = m(\mathbf{A}) + m(\mathbf{B});$$

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$, then $m(\mathbf{A}_n) \nearrow m(\mathbf{A})$

for all $\mathbf{A}, \mathbf{A}_n, \mathbf{B} \in \mathcal{M}$, $n \in N$.

State on MV-algebra $\mathcal{M}$

Embedding theorem

Let the system $(\mathcal{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ be the *MV*-algebra of IF-sets. Then $\mathcal{F} \subset \mathcal{M}$ and to each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists a MV-state $\overline{\mathbf{m}} : \mathcal{M} \to [0,1]$ such that $\overline{\mathbf{m}}|\mathcal{F} = \mathbf{m}$.

Really, if $(\mu_A, \nu_A) \in \mathcal{M}$, then $(\mu_A, 0_\Omega) \in \mathcal{F}$ and $(0_\Omega, 1_\Omega - \nu_A) \in \mathcal{F}$. It is reasonable to define

$$m\big((\mu_A, \nu_A)\big) = \mathbf{m}\big((\mu_A, 0_\Omega)\big) - \mathbf{m}\big((0_\Omega, 1_\Omega - \nu_A)\big).$$

It is not difficult to prove that $m : \mathcal{M} \to [0,1]$ is a state of MV-algebra $\mathcal{M}$ and $m|\mathcal{F} = \mathbf{m}$.

Intuitionistic Fuzzy Observable

Borel sets

Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a, b) = \{x \in R : a \leq x < b\}.$$

Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets.

IF-observable x

$x : \mathcal{B}(R) \to \mathcal{F}$

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \emptyset$ and $A, B \in \mathcal{B}(R)$, then
$x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and
$x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$ and $A_n, A \in \mathcal{B}(R)$, then $x(A_n) \nearrow x(A)$.

n-dimensional Intuitionistic Fuzzy Observable

$x : \mathcal{B}(R^n) \to \mathcal{F}$

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \emptyset$ and $A, B \in \mathcal{B}(R^n)$, then
$x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and
$x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$ and $A_n, A \in \mathcal{B}(R^n)$, then $x(A_n) \nearrow x(A)$.

Intuitionistic Fuzzy distribution function

$\mathbf{F} : R \longrightarrow [0,1]$

$$\mathbf{F}(t) = \mathbf{m}\big(x((-\infty, t))\big)$$

for each $t \in R$

$\mathbf{m} : \mathcal{F} \longrightarrow [0,1]$ be an IF-state
$x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be an IF-observable

Properties

- non-decreasing on R;
- left continuous in each point $t \in R$;
- $$\lim_{n \to -\infty} \mathbf{F}(t) = 0, \ \lim_{n \to \infty} \mathbf{F}(t) = 1$$

Moments of IF-observable

IF-expectation $\mathbf{E}(x)$

$$\mathbf{E}(x) = \int_R t \; d\mathbf{F}(t)$$

IF-dispersion $\mathbf{D}^2(x)$

$$\mathbf{D}^2(x) = \int_R t^2 \; d\mathbf{F}(t) - \left(\mathbf{E}(x)\right)^2 = \int_R (t - \mathbf{E}(x))^2 \; d\mathbf{F}(t).$$

Independence of IF-observables

IF-observables $x_1, x_2, ...x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ are independent if for n-dimensional IF-observable $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ there holds

$$\mathbf{m}\big(h_n(A_1 \times \ldots \times A_n)\big) = \mathbf{m}\big(x_1(A_1)\big) \cdot \ldots \cdot \mathbf{m}\big(x_n(A_n)\big)$$

for each $A_1, ..., A_n \in \mathcal{B}(R)$, where $\mathbf{m}$ be an IF-state.

Joint IF-observable of the IF-observables x, y

$h : \mathcal{B}(R^2) \to \mathcal{F}$

(i) $h(R^2) = (1_\Omega, 0_\Omega)$;

(ii) if $A, B \in \mathcal{B}(R^2)$ and $A \cap B = \emptyset$, then $h(A \cup B) = h(A) \oplus h(B)$ and $h(A) \odot h(B) = (0_\Omega, 1_\Omega)$;

(iii) if $A, A_1, \ldots \in \mathcal{B}(\mathbf{R}^2)$ and $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C) \bullet y(D)$ for each $C, D \in \mathcal{B}(R)$.

Product operation

$$\mathbf{A} \bullet \mathbf{B} = (\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B)$$

Theorem

To each two IF-observables $x, y : \mathcal{B}(\mathbf{R}) \to \mathcal{F}$ there exists their joint IF-observable.

Function of several IF-observables

$$y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$$

$$y_n = g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big)$$

for each $A \in \mathcal{B}(R)$.

Assumptions:

- $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ are IF-observables,
- h_n is their joint IF-observable,
- $g_n : R^n \to R$ is a Borel measurable function.

Function of several IF-observables

$$y_n = h_n \circ g_n^{-1}$$

Examples

- $y_n = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum_{i=1}^{n} x_i - a\Big)$,

  $g_n(u_1, \dots, u_n) = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum_{i=1}^{n} u_i - a\Big)$;
- $y_n = \frac{1}{n}\sum_{i=1}^{n} x_i$, $g_n(u_1, \dots, u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i$;
- $y_n = \frac{1}{n}\sum_{i=1}^{n} (x_i - \mathbf{E}(x_i))$,

  $g_n(u_1, \dots, u_n) = \frac{1}{n}\sum_{i=1}^{n} (u_i - \mathbf{E}(x_i))$;
- $y_n = \frac{1}{a_n}\big(\max(x_1, \dots, x_n) - b_n\big)$,
  $g_n(u_1, \dots, u_n) = \frac{1}{a_n}\big(\max(u_1, \dots, u_n) - b_n\big)$.

Convergence in distribution

Let $(y_n)_n$ be a sequence of IF-observables in the IF-space $(\mathcal{F}, \mathbf{m})$. We say that $(y_n)_n$ converges in distribution to a function $\Psi : R \longrightarrow [0,1]$, if for each $t \in R$

$$\lim_{n\to\infty} \mathbf{m}(y_n((-\infty, t))) = \Psi(t).$$

Central limit theorem

Let $(\mathcal{F}, \mathbf{m})$ be an IF-space, $(x_n)_n$ be a sequence of independent IF-observables with the same distribution $\mathbf{m}_x$ and such that $\mathbf{D}^2(x_n) = \sigma^2, \mathbf{E}(x_n) = a, (n = 1, 2, \dots)$ and $y_n = \frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} x_i - a\right)$. Then for all $t \in R$

$$\lim_{n\to\infty} \mathbf{m}(y_n((-\infty, t))) = \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{u^2}{2}}\, du,$$

i.e. the sequence $(y_n)_n$ converges in distribution to $\Phi : R \longrightarrow [0,1]$.

Almost everywhere convergence

Let $(y_n)_n$ be a sequence of IF-observables on an IF-space $(\mathcal{F}, \mathbf{m})$. We say that $(y_n)_n$ converges to 0 $\mathbf{m}$-almost everywhere, if

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \mathbf{m}\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1.$$

Almost everywhere convergence

Strong law of large numbers

Let $(\mathcal{F}, \mathbf{m})$ be an IF-state space, $(x_n)_n$ be a sequence of independent IF-observables such that $\mathbf{D}^2(x_n)$ exists for every $n \in N$ and $\sum_{n=1}^{\infty} \frac{\mathbf{D}^2(x_n)}{n^2} < \infty$. Then $(y_n)_n$ converges $\mathbf{m}$-almost everywhere to 0, i.e.

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \mathbf{m}\Big(\bigwedge_{n=k}^{k+i} y_n\Big(-\frac{1}{p}, \frac{1}{p}\Big)\Big) = 1,$$

where $y_n = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mathbf{E}(x_i))$, $n \in \mathbf{N}$.

Convergence in measure

Let $(y_n)_n$ be a sequence of IF-observables in the IF-space $(\mathcal{F}, \mathbf{m})$. We say that $(y_n)_n$ converges in measure $\mathbf{m}$ to 0, if for each $\varepsilon > 0, \varepsilon \in R$

$$\lim_{n \to \infty} \mathbf{m}(y_n((-\varepsilon, \varepsilon))) = 1.$$

Weak law of large numbers

Let $(\mathcal{F}, \mathbf{m})$ be an IF-space, $(x_n)_n$ be a sequence of independent IF-observables with the same distribution $\mathbf{m}_x$ and such that $\mathbf{E}(x_n) = a, (n = 1, 2, \dots)$ and $y_n = \frac{1}{n} \sum_{i=1}^{n} x_i - a$. Then for each $\varepsilon > 0, \varepsilon \in R$

$$\lim_{n \to \infty} \mathbf{m}(y_n((-\varepsilon, \varepsilon))) = 1.$$

Almost uniform convergence

Let $(\mathcal{F}, \mathbf{m})$ be an IF-space with an IF-state $\mathbf{m}$. The sequence $(x_n)_1^\infty$ of IF-observables converges $\mathbf{m}$-almost uniformly to 0 if to every $\alpha > 0$ there exists an IF-set $\mathbf{A} \in \mathcal{F}$ such that $\mathbf{m}(\mathbf{A}) > 1 - \alpha$ and such that to every $\beta > 0$ there exists k such that $\mathbf{A} \leq x_n\big((-\beta, \beta)\big)$ for every $n \geq k$.

Egorov's Theorem

Let $(\mathcal{F}, \mathbf{m})$ be an IF-space with an IF-state $\mathbf{m}$. If a sequence $(x_n)_1^\infty$ of IF-observables converges $\mathbf{m}$-almost everywhere to 0, then the sequence $(x_n)_1^\infty$ converges $\mathbf{m}$-almost uniformly to 0.

Interval Valued Intuitionistic Fuzzy Sets

G. Gargov in 1989

- generalization of the IF-sets and IVF-sets

Interval valued intuitionistic fuzzy set A on a space E

$$A = \{\langle x, M_A(x), N_A(x)\rangle | x \in E\},$$

where

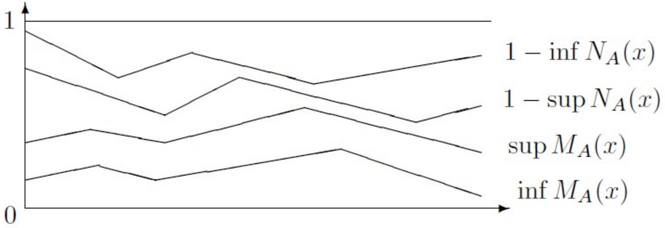
$$M_A(x) \subset [0,1] \text{ and } N_A(x) \subset [0,1]$$

are intervals and for all $x \in E$:

$$\sup M_A(x) + \sup N_A(x) \leq 1.$$

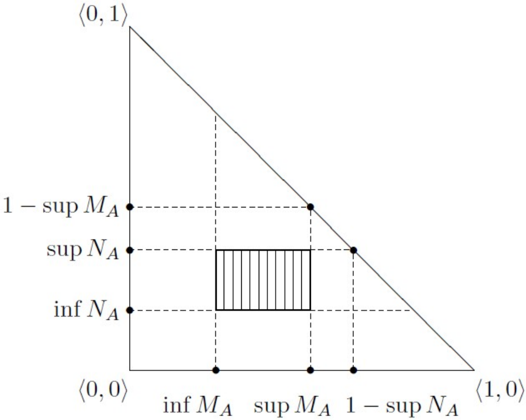
Interval Valued Intuitionistic Fuzzy Sets

First geometrical interpretation



Interval Valued Intuitionistic Fuzzy Sets

Second geometrical interpretation



Thank you very much for your attention!