Constructing the free orthomodular poset over an orthoposet

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Motivation

Denote BPI the category of bounded posets with an involution, and OMP the category of orthomodular posets.

Theorem (Jenča, 2022)

The forgetful functor from the category OMP *into the category* BPI *has a monadic left adjoint functor.*

The proof of the above theorem is *non-constructive*.
Our aim is to describe this adjoint functor in an *explicit* way.

- Jenča, G., *Orthomodular posets are algebras over bounded posets with involution*, Soft Computing 26 (2022), 491–498.

Categories and functors

- Category is a collection of objects and morphisms (transformations of objects). Morphisms can be composed.
- Functor from a category $\mathcal{C}$ into a category $\mathcal{D}$ maps objects/morphisms of $\mathcal{C}$ into objects/morphisms of $\mathcal{D}$ so that this mapping *plays well* with the definitions of both categories.

Bounded posets with an involution

Let $(P, \leq)$ be a poset. An unary operation $' \colon P \to P$ is an involution if

1. $(x')' = x$ for all $x \in P$,
2. $x \leq y$ iff $y' \leq x'$ for all $x, y \in P$.

A structure $(P, \leq, ', 0, 1)$ is a bounded poset with an involution (BPI) if $(P, \leq)$ is a poset with involution $'$, bottom element 0, and top element 1.

Let P, Q be BPIs. A map $f \colon P \to Q$ is a BPI-morphism if

1. $x \leq y \Rightarrow f(x) \leq f(y)$ for all $x, y \in P$,
2. $f(x') = (f(x))'$ for all $x \in P$,
3. $f(0) = 0,\ f(1) = 1$.

Denote BPI the category where objects are bounded posets with an involution and morphisms are BPI-morphisms.

Orthoposets and orthomodular posets

An orthoposet (OP) is a BPI P such that

$x \wedge x' = 0$ for all $x \in P$.

Denote OP the category where objects are orthoposets and morphisms are BPI-morphisms.

An orthomodular poset (OMP) is an OP P such that

1. for all $x, y \in P$, if $x \leq y'$ then $x \vee y$ exists in P,
2. for all $x, y \in P$, if $x \leq y'$ then $x \vee (x \vee y)' = y$.

Let P, Q be OMPs. A map $f \colon P \to Q$ is an OMP-morphism if it is a BPI-morphism and

for all $x, y \in P$, if $x \leq y'$ then $f(x \vee y) = f(x) \vee f(y)$.

Denote OMP the category where objects are OMPs and morphisms are OMP-morphisms.

Examples of orthomodular posets

- Every Boolean algebra is an OMP.
- There exist OMPs which are not Boolean algebras.
- There exist OMPs which are not lattices.
- There exist OMPs which are non-distributive lattices.
- Every OMP can be obtained as a pasting of Boolean algebras.

Natural transformations and adjunctions

Let $\mathcal{C}, \mathcal{D}$ be categories and let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$, $G \colon \mathcal{C} \longrightarrow \mathcal{D}$ be functors.
A natural transformation $\alpha \colon F \Longrightarrow G$ maps every object x of the category $\mathcal{C}$ into a morphism α_x of the category $\mathcal{D}$ so that it *plays well* with both functors.

Namely, for every morphism $f \colon X \to Y$ of $\mathcal{C}$, the following holds in $\mathcal{D}$:

$$\begin{array}{ccc} FX & \xrightarrow{F} & FY \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ GX & \xrightarrow[G]{} & GY \end{array}$$

Let $\mathcal{C}, \mathcal{D}$ be categories and let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$, $G \colon \mathcal{D} \longrightarrow \mathcal{C}$ be functors.
An adjunction $F \dashv G$ is a pair of natural transformations $\eta \colon 1_{\mathcal{C}} \Longrightarrow GF$, $\varepsilon \colon FG \Longrightarrow 1_{\mathcal{D}}$.

Forgetful and free functors

An object/morphism in category OMP can be viewed as an object/morphism in category BPI.

This can be expressed as a functor $U\colon \mathrm{OMP} \longrightarrow \mathrm{BPI}$, defined by $UP = P$ and $Uf = f$, for every OMP P and every OMP-morphism f.

Such functor U is called "forgetful" since it forgets some structure or properties. If there exists an adjunction $F \dashv U$ then functor F is called "free".

Our aim is to construct the free functor $F\colon \mathrm{BPI} \longrightarrow \mathrm{OMP}$.

We will do it in two steps. We construct free functors $F_1\colon \mathrm{BPI} \longrightarrow \mathrm{OP}$, $F_2\colon \mathrm{OP} \longrightarrow \mathrm{OMP}$, and then we take $F = F_2 F_1$.

From BPIs to OPs

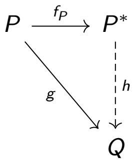
Let P be a BPI. Define a function $f_P\colon P \to P$ by

$$f_P(x) = \begin{cases} 0 & x \leq x' \\ 1 & x' \leq x \\ x & \text{otherwise} \end{cases}$$

Denote $P^* = f_P[P]$. Then P^* is an OP and $f_P\colon P \to P^*$ is a BPI-morphism.

Theorem

For every OP Q and every BPI-morphism $g\colon P \to Q$ there exists a unique OP-morphism $h\colon P^* \to Q$ such that $g = h \circ f_P$.



This is sufficient to define the free functor $F_1\colon \mathrm{BPI} \longrightarrow \mathrm{OP}$.

Namely, $F_1 P = P^*$ for every BPI P, $F_1 f = f_Q \circ f \circ i_P$ for every BPI-morphism $f\colon P \to Q$, where $i_P\colon P^* \to P$ is the inclusion map.

From OPs to OMPs

Let P be an OP.

Denote $\mathcal{L}_{\mathcal{P}}$ the language consisting of an unary operation symbol $'$, a binary operation symbol $+$, and a constant c_x for every element $x \in P$.

Denote $\mathcal{T}_P$ the set of all terms in the language $\mathcal{L}_P$.

Given an OMP Q and a BPI-morphism $f\colon P \to Q$, let val_f be a function with the smallest domain from a subset of $\mathcal{T}_P$ into Q such that:

1. $\mathrm{val}_f(c_x) = f(x)$, for every $x \in P$,
2. $\mathrm{val}_f(\tau') = \mathrm{val}_f(\tau)'$, for every $\tau \in \mathrm{dom}(\mathrm{val}_f)$,
3. $\mathrm{val}_f(\tau + \sigma) = \mathrm{val}_f(\tau) \vee \mathrm{val}_f(\sigma)$, for every $\tau, \sigma \in \mathrm{dom}(\mathrm{val}_f)$ such that $\mathrm{val}_f(\tau) \leq \mathrm{val}_f(\sigma)'$.

From OPs to OMPs

We say that a term $\tau \in \mathcal{T}_P$ is well-formed if $\tau \in \mathrm{dom}(\mathrm{val}_f)$ for every OMP Q and every BPI-morphism $f\colon P \to Q$. Denote $\mathcal{W}_P$ the set of all well-formed terms $\tau \in \mathcal{T}_P$.

For $\tau, \sigma \in \mathcal{W}_P$, let $\tau \preceq \sigma$ iff $\mathrm{val}_f(\tau) \leq \mathrm{val}_f(\sigma)$ holds for every OMP Q and every BPI-morphism $f\colon P \to Q$.

Let $\tau \approx \sigma$ if $\tau \preceq \sigma$ and $\sigma \preceq \tau$. Then $\preceq$ is a preorder and $\approx$ is an equivalence relation.

Let $\mathcal{F}_P$ be the quotient $\mathcal{W}_P/{\approx}$, that is, the elements of $\mathcal{F}_P$ are the equivalence classes $[\tau]_\approx = \{\sigma \in \mathcal{W}_P \colon \tau \approx \sigma\}$. The set $\mathcal{F}_P$ is partially ordered by the relation $\leq$ where $[\tau]_\approx \leq [\sigma]_\approx$ iff $\tau \preceq \sigma$.

From OPs to OMPs

Fact

Let $\tau, \sigma \in \mathcal{W}_P$.

1. $\tau \approx \sigma$ *iff* $\tau' \approx \sigma'$,
2. *if* $\tau \preceq \sigma'$ *then* $\tau + \sigma \in \mathcal{W}_P$ *and* $[\tau + \sigma]_\approx = [\tau]_\approx \vee [\sigma]_\approx$.

This allows us to define $([\tau]_\approx)' = [\tau']_\approx$. Denote $\mathbf{0} = [0]_\approx$, $\mathbf{1} = [1]_\approx$.

Fact

1. *The structure* $P^* = (\mathcal{F}_P, \leq, ', \mathbf{0}, \mathbf{1})$ *is an OMP.*
2. *Mapping* $f_P \colon P \to P^*$, *defined by* $f_P(x) = [c_x]_\approx$ *for all* $x \in P$, *is a BPI-morphism.*

Theorem

For every OMP Q and for every BPI-morphism $g \colon P \to Q$ there exists a unique OMP-morphism $h \colon P^* \to Q$ such that $g = h \circ f_P$.

This is again sufficient to define the free functor $F_2 \colon \mathrm{OP} \to \mathrm{OMP}$.

From OPs to OMPs

Problems

1. Characterize terms $\tau \in \mathcal{W}_P$.
2. For $\tau, \sigma \in \mathcal{W}_P$, characterize $\tau \preceq \sigma$.

Conjecture

Let P be an OP. Then for every $\tau \in \mathcal{W}_P$ there exists $\sigma \in \mathcal{W}_P$ such that $\tau \approx \sigma$ and for every $x \in P$, constant c_x occurs in σ at most once.

Conjecture

Let P be an OP. Then $\tau \preceq \sigma$ iff for every BPI-morphism $f\colon P \to \{0, 1\}$, $\mathrm{val}_f(\tau) \leq \mathrm{val}_f(\sigma)$.