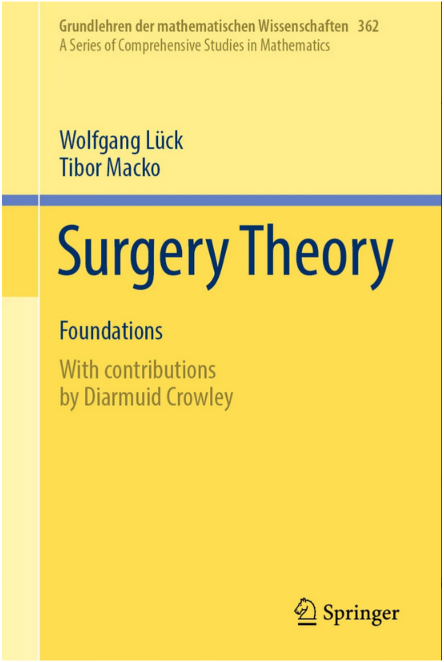
Wolfgang Lück, Tibor Macko: Surgery theory: Foundations with contributions by Diarmuid Crowley

Springer, Grundlehren der mathematischen Wissenschaften, Vol. 362

Tibor Macko

Comenius University and
Slovak Academy of Sciences
Bratislava
www.mat.savba.sk/~macko

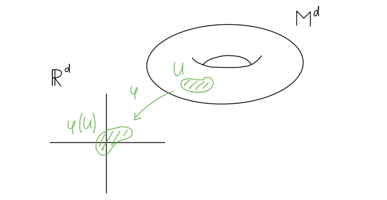
Smolenice
October 2024



Wolfgang Lück
University of Bonn
C3-Professor of mathematics
PhD Göttingen 1984
Lexington, Mainz, Münster
Leibniz Prize
ERC Advanced grant

Background 1

Manifold M



Versions: $\mathrm{CAT} = \mathrm{TOP}, \mathrm{PL}, \mathrm{DIFF}$ and mostly assume compact (closed)
DIFF: $(M, \mathcal{F}_M)$, $\mathcal{F}_M = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ s.t. $\varphi_\beta \circ \varphi_\alpha^{-1} \in C^\infty$

Examples

$$\mathbb{R}^d, S^d, S^m \times S^n, T^n, F_g^+, F_g^-, \text{Lie group } G, G/H, \ldots$$

Applications

mathematical physics, dynamical systems, algebraic geometry, ...

Background 2

Important prior results

- Novikov (1955) - impossible to classify all manifolds of a fixed dim
- Milnor (1956) - discovery of exotic spheres
- Smale (1958) - generalized Poincaré conjecture in dim ≥ 5

If $d \geq 4$, then for any f.p. group G there exists M^d s.t. $\pi_1(M^d) \cong G$.

$\Sigma^7 \ncong_{\mathrm{DIFF}} S^7$, but $\Sigma^7 \cong_{\mathrm{TOP}} S^7$

If $M \simeq S^n$, then $M \cong_{\mathrm{TOP}} S^n$.

$L^3(7;1,1) \simeq L^3(7;2,1)$, but $L^3(7;1,1) \ncong_{\mathrm{TOP}} L^3(7;2,1)$

$X \simeq Y$ if $\exists\, f\colon X \to Y$, $g\colon Y \to X$ s.t. $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$

Surgery theory

The structure set of X

$$\mathcal{S}^{\mathrm{CAT}}(X) := \{f : M \xrightarrow{\simeq} X\}/\sim$$

The surgery exact sequence (Browder-Novikov-Sullivan-Wall)

For an n-manifold X with $n \geq 5$ nad $\pi = \pi_1(X)$ we have

$$\cdots \to \mathcal{N}_{\partial}^{\mathrm{CAT}}(X \times I) \xrightarrow{\sigma_{n+1}} L_{n+1}(\mathbb{Z}\pi) \xrightarrow{\rho_n} \mathcal{S}^{\mathrm{CAT}}(X) \xrightarrow{\eta_n} \mathcal{N}^{\mathrm{CAT}}(X) \xrightarrow{\sigma_n} L_n(\mathbb{Z}\pi).$$

Explanation

- $\mathcal{N}^{\mathrm{CAT}}(X)$ - normal cobordism - generalized cohomology theory
- $L_n(\mathbb{Z}\pi)$ - Witt group of (automorphisms of) quadratic forms
- σ - surgery obstruction map

The surgery exact sequence (Thm 11.22 (page 413))

Normal invariants - via cobordisms

$$((f_0, \overline{f}_0) \colon M_0 \to X) \sim ((f_1, \overline{f}_1) \colon M_1 \to X)$$

if exists

$$(F, \overline{F}) \colon (W; M_0, M_1) \to (X \times [0,1], X \times 0, X \times 1)$$

s.t. for $j = 0, 1$

$$(F, \overline{F}) \circ i_j = (f_j, \overline{f}_j).$$

L-groups

$L_n(\mathbb{Z}) = \mathbb{Z}$ (signature), $0, \mathbb{Z}/2$ (Arf invariant), 0

The surgery obstruction map when $\pi_1(X) = 0$

$\sigma \colon N^{\mathrm{CAT}}(X) \to L_{4k}(\mathbb{Z}), \quad \sigma(f, \overline{f}) := (1/8)(\mathbf{sign}(M) - \mathbf{sign}(X)).$

Results

Homotopy spheres (Chapter 12)

$\mathcal{S}^{\mathrm{DIFF}}(S^n) = \Theta_n$ is a finite group, in terms of $J_k \colon \pi_k SO \to \pi_k^S$, $k = n, n+1$.

Calculations in the TOP category (Chapters 18,19)

$$\mathcal{S}^{\mathrm{TOP}}(X) \quad \text{for} \quad X = S^n, S^k \times S^l, \mathbb{C}P^n, \mathbb{R}P^n, L_N^{2d-1}, T^n, \ldots$$

When $\pi_1(X)$ is finite calculations involve representation theory of $\pi_1(X)$.

The Borel conjecture (Chapter 19) and Lück: IC, 2025?

$$\mathcal{S}^{\mathrm{TOP}}(BG) \cong \{\mathrm{id}_{BG}\}?$$

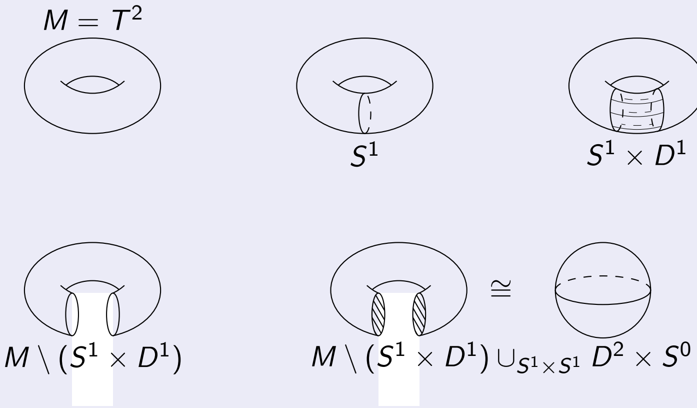
Proofs involve the so-called assembly maps.

Table of contents

Table of contents

Geometric surgery

Figure (surgery step)



Examples

Milnor's exotic spheres in $\mathcal{S}^{\mathrm{DIFF}}(S^7)$

$S^3 \to \Sigma \to S^4$ bundles are classified by $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.
For $(k, l) \in \pi_3(SO(4))$ take $\Sigma = \Sigma(k, l)$ and note $\Sigma = \partial W$ for smooth W.
If $\exists\, h \colon \Sigma \xrightarrow{\cong_{\mathrm{DIFF}}} S^7$ then $M := W \cup_h D^8$ is a smooth manifold.

But $\mathbb{Z} \in \mathbf{sign}(M) = \langle L(M), [M] \rangle \notin \mathbb{Z}$ by the Hirzebruch signature theorem.

Examples in $\mathcal{S}^{\mathrm{TOP}}(S^p \times S^q) \cong L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z})$ via splitting invariants

Let $h \colon M \xrightarrow{\simeq} S^p \times S^q$.
Make $h \pitchfork S^p \times \{*\}$ and consider $g \colon N = h^{-1}(S^p \times \{*\}) \to S^p \times \{*\}$.

Then $s_{4i}(h) = (1/8)(\mathbf{sign}(N) - 0)$ detects $\mathcal{S}^{\mathrm{TOP}}(S^{4i} \times S^q) \cong \mathbb{Z} \oplus L_q(\mathbb{Z})$.

Algebraic surgery (Ranicki)

Homological algebra

R - ring, M - R-module, $\mathrm{Hom}_R(M, —)$ is not a right exact functor
$\rightsquigarrow \mathrm{Ext}^n_R(M,N) := H^n(\mathrm{Hom}_R(P_*, N))$, with $P_* \to M$ proj. R-res. of M

Chain complex

$$C_* = \cdots \to C_{r+1} \xrightarrow{c_{r+1}} C_r \xrightarrow{c_r} C_{r-1} \to \cdots$$

A symmetric form on an R-module

$$\varphi \in \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}, P^* \otimes_R P^*)$$

An n-dim symmetric structure on an R-chain complex

$$[\varphi] \in H^n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, C \otimes_R C)) \quad \text{with} \quad W \to \mathbb{Z} \quad \text{free } \mathbb{Z}[\mathbb{Z}/2]-\text{res.}$$

Assembly maps

Geometric surgery exact seq. $\rightsquigarrow$ Algebraic surgery exact seq.

$$\begin{array}{ccccccc}
L_{n+1}(\mathbb{Z}\pi) & \xrightarrow{\rho_n} & \mathcal{S}^{\mathrm{TOP}}(X) & \xrightarrow{\eta_n} & \mathcal{N}^{\mathrm{TOP}}(X) & \xrightarrow{\sigma_n} & L_n(\mathbb{Z}\pi) \\
\downarrow = & & \cong \downarrow \mathbf{qsign}_X & & \cong \downarrow \mathbf{qsign}_X & & \downarrow = \\
L_{n+1}(\mathbb{Z}\pi) & \xrightarrow{\partial_n} & \mathbb{S}_{n+1}(X) & \xrightarrow{i_n} & H_n(X; \mathbf{L}_\bullet\langle 1\rangle) & \xrightarrow{\mathrm{asmb}} & L_n(\mathbb{Z}\pi)
\end{array}$$

The Farrell-Jones conjecture for torsion-free group G

$\mathrm{asmb}\colon H_n(BG; \mathbf{L}_\bullet) \to L_n(\mathbb{Z}G)$ is an isomorphism for all $n \in \mathbb{Z}$.

Future

Kervaire invariant 1 problem

Determine $\theta \colon \mathcal{N}^{\mathrm{DIFF}}(S^n) \to \mathbb{Z}/2$ when $n = 126$.

The Borel conjecture

$$\mathcal{S}^{\mathrm{TOP}}(BG) \cong \{\mathrm{id}_{BG}\}?$$

More calculations in TOP

$$\mathcal{S}^{\mathrm{TOP}}(M) \cong \quad ? \quad \text{when} \quad \pi_1(M) \quad \text{is infinite, but contains torsion}$$

More calculations in DIFF

$$\mathcal{S}^{\mathrm{DIFF}}(M) \cong \quad ? \quad \text{when} \quad M \neq S^n, S^p \times S^q, \mathbb{C}P^n, \mathbb{R}P^n, \ldots$$

Combine with other tools

algebraic K-theory, smoothing theory, (higher) cobordism categories, h-principles, homotopy theory, manifold calculus, geometric group theory...

Basic literature

- Kervaire, M. and Milnor, J.: *Groups of homotopy spheres. I.*, 1963.
- Browder, W.: *Surgery on simply-connected manifolds*, Springer, 1971
- Wall, C.T.C.: *Surgery on compact manifolds*, LMS, 1970
- Ranicki, A.: *Geometric and algebraic surgery*, CUP, 2003.
- Chang, S. and Weinberger, S.: *A course in surgery theory*, PUP, 2021.